Cosmology: Example Sheet 1 Michaelmas Term
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1. Olbers paradox is the observation that, in an infinitely large and infinitely old universe, the sky at night will not be dark.

   Consider stars, each of radius R, distributed evenly, but randomly, throughout an infinite universe with average separation ℓ. Show that, along any line of sight, stars will be observed at an average distance

   $$d \sim \frac{\ell^3}{R^2}\,.$$

   Each star shines with luminosity L and, at a distance r, produces an energy flux per unit area $\Phi = L/(4\pi r^2)$. Show that the total energy flux per unit area on Earth is $\Phi \sim L/(\pi R^2)$. Show further that this equals the energy flux per unit area that the Earth would receive if each point on the sky were as bright as the Sun.

   In our Universe, the average mass density corresponds to about 1 hydrogen atom per cubic metre, with the mass of a hydrogen atom given by $m_H \approx 1.7 \times 10^{-27}\,\mathrm{kg}$. Assuming that this mass primarily lies in stars similar to the Sun, with mass $M_\odot \approx 2\times 10^{30}\,\mathrm{kg}$ and radius $R_\odot \approx 7\times 10^{8}\,\mathrm{m}$, show that $d \approx 10^{39}\,\mathrm{m}$.

   The Universe expands with Hubble constant $H_0 \approx 10^{-18}\,\mathrm{s}^{-1}$. Explain why we should not expect the night sky to be bright.

2. A spherical cloud of mass M and initial radius R contains material with uniform density and zero pressure. The shell undergoes gravitational collapse, with the radius $r(t)$ governed by Newtons law

   $$\ddot{r} = -\frac{G\,M}{r^2}\,.$$

   Show that the radius obeys the parametric solution

   $$r(\theta) = R\,\cos^2\theta \quad \theta + \frac{1}{2}\sin 2\theta = A\,t$$

   for some suitable constant A. Sketch the behaviour of $r(t)$ as the cloud collapses from $r = R$ to $r = 0$. Show that the cloud collapses completely in time $t_{\mathrm{col}} = \sqrt{3\pi/(32\,G\,\rho_0)}$ where ρ_0 is the initial mass density.

   Give an order of magnitude estimate (in years) for the collapse timescale $\sim (G\rho_0)^{-1/2}$ of our galaxy, assuming that a typical star has one solar mass with an interstellar separation of one parsec.

   [$M_\odot \approx 2\times 10^{30}\,\mathrm{kg}$ $1\,\mathrm{pc} \approx 3\times 10^{16}\,\mathrm{m}$, $1\,\mathrm{yr} \approx 3\times 10^{7}\,\mathrm{s}$.]

3. Explain the difference between the *particle horizon* and the *cosmological event horizon* in an expanding universe. Which horizon exists when $a(t) = (t/t_0)^n$? Which horizon exists when $a(t) = e^{Ht}$? Are there FRW metrics that exhibit neither horizon? What about both?

4. Consider an FRW universe, dominated by a perfect fluid with pressure $P = \omega\,\rho$. Define the time-dependent density parameter

   $$\Omega(t) = \frac{\rho(t)}{\rho_{\mathrm{crit}}(t)}$$

where $\rho_{\text{crit}} = 3H^2c^2/(8\pi G)$. Use the continuity equation and acceleration equation to show that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\log a} = (1+3\omega)\Omega(\Omega-1)$$

Hence, show that a flat universe is unstable for $\omega > -1/3$ and stable for $\omega < -1/3$.

5. Consider an FRW universe with $k = 0, \pm 1$, dominated by a fluid with equation of state $P = \omega\rho$. Define dimensionless conformal time

$$\tilde{\tau}(t) = \frac{c}{R}\int^t \frac{\mathrm{d}t'}{a(t')} \,.$$

Show that the Friedmann equation can be written as

$$h^2 + k = \frac{8\pi G R^2}{3c^4}\rho\, a^2 \,,$$

where $h = a'/a$ and $a' = \mathrm{d}a/\mathrm{d}\tilde{\tau}$. Use the Raychaudhuri (acceleration) equation to show that

$$2h' + (1+3\omega)(h^2+k) = 0 \,.$$

Introduce a new variable $y = a^{(1+3\omega)/2}$. Show that

$$y'' = -\frac{1}{4}(1+3\omega)^2\, k\, y \,.$$

Consider a radiation-dominated universe with energy density $\rho = \rho_0/a^4$ and positive curvature, $k = +1$. Find the solution $a(\tilde{\tau})$ and $t(\tilde{\tau})$ in terms of ρ_0, subject to the requirement $a(0) = t(0) = 0$. Sketch the graph of $a(t)$ against t. Find the total time duration from Big Bang to Big Crunch as a function of ρ_0.

6. Consider a flat universe containing matter and radiation. Use conformal time

$$\tau(t) = \int^t \frac{\mathrm{d}t'}{a(t')}$$

to show that the Friedmann equation can be written as

$$\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2 = A(a + a_{\text{eq}}) \,,$$

where a_{eq} is the scale factor when the energy densities of matter and radiation are equal. Show that the solution is given by

$$a(\tau) = \frac{1}{4}A\,\tau^2 + B\,\tau \,,$$

for some B. Determine $t(\tau)$ and use this to find the asymptotic behaviour of $a(t)$ when $a \gg a_{\text{eq}}$ and $a \ll a_{\text{eq}}$.

7. The metric of $d = 4+1$ dimensional Minkowski spacetime is given by

$$\mathrm{d}s^2 = -c^2\mathrm{d}T^2 + \sum_{i=1}^{4}\left(\mathrm{d}X^i\right)^2 \,.$$

de Sitter spacetime in $d = 3+1$ dimensions can be viewed as timelike hyperbola

$$-c^2T^2 + \sum_{i=1}^{4}\left(X^i\right)^2 = R^2 \,,$$

with R constant.

i) Confirm that the constraint can be solved by

$$c\,T = R\sinh\left(\frac{c\,t}{R}\right) \quad \text{and} \quad X^i = R\cosh\left(\frac{c\,t}{R}\right)\, y^i\,,$$

where y^i obey $\sum_{i=1}^4 \left(y^i\right)^2 = 1$ and hence parameterise a 3-dimensional sphere. Show that the induced $d = 3+1$ metric is de Sitter spacetime with $k = +1$,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + R^2\cosh^2\left(\frac{c\,t}{R}\right)\mathrm{d}\Omega_3^2\,,$$

where $\mathrm{d}\Omega_3^2$ is the metric on 3-sphere of unit radius.

ii*) Confirm that the constraint can alternatively be solved by

$$c\,T = R\,\sinh\left(\frac{c\,t}{R}\right) + \frac{r^2}{2\,R}e^{\frac{c\,t}{R}}\,, \quad X^p = e^{\frac{c\,t}{R}}x^p \quad \text{and} \quad X^4 = R\cosh\left(\frac{c\,t}{R}\right) - \frac{r^2}{2\,R}e^{\frac{c\,t}{R}}\,,$$

where $x^p\,, p = 1, 2, 3,$ are constrained and $r^2 = \sum_{p=1}^3 (x^p)^2$. Show that in these coordinates, the induced metric is de Sitter spacetime with k $= 0$,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + e^{\frac{2\,c\,t}{R}}\sum_{p=1}^{3}\mathrm{d}x^p\mathrm{d}x^p\,.$$