Cosmology: Example Sheet 2 Michaelmas Term
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1. In special and general relativity, the energy density ρ and pressure P of a perfect fluid combine to form the *energy-momentum* tensor. In the rest frame of the fluid, this is given by

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} .$$

After a Lorentz boost by velocity v in the $x-$direction, the energy-momentum tensor transforms to

$$\widetilde{T}^{\mu\nu} = \Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\sigma}T^{\lambda\sigma} \quad \text{with} \quad \Lambda^{\mu}{}_{\lambda} = \left[\begin{array}{cc|cc} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} .$$

What equation of state $P = \omega\,\rho$ gives rise to a fluid that looks the same to all inertial observers?

2. The *deceleration parameter* is defined by

$$q(t) \equiv -\frac{a\ddot{a}}{\dot{a}^2} .$$

Compute q for a flat universe dominated by a single fluid with equation of state $P = \omega\rho$.

For a universe with arbitrary curvature, filled with matter, radiation and a cosmological constant, show that today

$$q_0 = \frac{1}{2}\Omega_{\rm m} + \Omega_{\rm rad} - \Omega_\Lambda .$$

What is the deceleration parameter for our universe?

Show that the Taylor expansion of the scale factor about the present day (with $a(t_0) = 1$) can be written as

$$\frac{1}{1+z} = a(t) = 1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2(t-t_0)^2 + \dots .$$

Hence show that the light from a nearby galaxy at redshift $z \ll 1$ was emitted at time

$$t_0 - t = H_0^{-1}z - \frac{1}{2}H_0^{-1}(2+q_0)z^2 + \dots .$$

3. A flat universe, containing matter and a cosmological constant, expands according to the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left(\rho_\Lambda + \frac{\rho_0}{a^3}\right) .$$

Solve this to find

$$a(t) = \beta(\sinh \alpha t)^{2/3} \tag{1}$$

with α and β constant. [Hint: you may find the substitution $b = a^{3/2}$ helpful.] Verify the expected asymptotic behaviour for early and late times.

Our universe today is flat, with $\Omega_{\rm m} \approx 0.3$ and $\Omega_\Lambda \approx 0.7$ and $H_0^{-1} \approx 14\,{\rm Gyr}$.

Determine:

- The age of the universe.
- The age at which the expansion of the universe first started to accelerate.
- The age at which the energy density in matter and the cosmological constant were equal.

A cosmologically ignorant civilisation measures $H_0^{-1} \approx 14\,\mathrm{Gyr}$ but refuses to countenance the possibility of a cosmological constant. How old would they believe their universe to be if they assumed it was flat with $\Omega_\mathrm{m} = 1$?

4. The *jerk* is defined as

$$J(t) = \frac{\dddot{a}\, a^2}{\dot{a}^3} .$$

14 billion years of cosmic expansion can be roughly characterised as: the jerk is one. To see this, consider the Friedmann equation with matter, a cosmological constant and curvature. Show that the curvature can be written as

$$\frac{k\, c^2}{R^2} = a^2 H^2 (J - 1) .$$

Confirm that the solution (1) indeed has unit jerk.

[Fun fact: the 4^{th}, 5^{th} and 6^{th} derivatives of the scale factor are called *snap*, *crackle* and *pop* respectively.]

5. A flat universe, containing radiation and a cosmological constant, expands according to the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left(\rho_\Lambda + \frac{\rho_0}{a^4}\right) .$$

Solve for $a(t)$. Verify the expected asymptotic behaviour for early and late times.

6. Consider a collection of particles, each of which interacts through the potential

$$V(\mathbf{x}_i - \mathbf{x}_j) = \alpha\, |\mathbf{x}_i - \mathbf{x}_j|^n .$$

Prove the virial theorem,

$$2\overline{T} = n\, \overline{V} ,$$

where $\overline{T}$ is the time-averaged kinetic energy and $\overline{V}$ the time-averaged total potential energy.

7. The equations of motion for inflation can be written as

$$H^2 = \frac{1}{3\, M_{\mathrm{pl}}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \tag{2}$$

and

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 , \tag{3}$$

where $M_{\mathrm{pl}}^2 = c^2/(8\pi G)$ is related to the *reduced Planck mass*. Consider the potential

$$V(\phi) = V_0\, e^{-\frac{\lambda\phi}{M_{\mathrm{pl}}}}$$

Show that the inflationary equations have the exact solution

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{\lambda^2}} \quad \text{and} \quad \phi(t) = \phi_0 + \alpha \log t\,,$$

for some choice of ϕ_0 and α.

For what values of λ does inflation occur? Define the pressure-energy density ratio $\omega_\phi = P_\phi/\rho_\phi$. For what range of ω_ϕ does inflation occur?

8. Assume that an inflationary solution $\phi(t)$ is monotonic and view the Hubble parameter $H(t)$ as a function $H(\phi(t))$. Show that the equations of inflation (2) and (3) can be written in Hamilton-Jacobi form

$$\dot{\phi} = -2M_{\rm pl}^2 \frac{\partial H}{\partial \phi}$$

and

$$V(\phi) = 3M_{\rm pl}^2 H^2 - 2M_{\rm pl}^4 \left(\frac{\partial H}{\partial \phi}\right)^2 .$$

[Hint: you might start by differentiating the Friedmann equation (2) with respect to time.]

The Hamilton-Jacobi formalism can be used to construct exact solutions in which one specifies a choice of $H(\phi)$ and then works backwards to read off the corresponding potential $V(\phi)$, the scalar evolution $\phi(t)$ and, hence the scale factor $a(t)$. Find these three quantities for

$$H(\phi) = \exp\left(-\sqrt{\frac{\alpha}{2}}\frac{\phi}{M_{\rm pl}}\right) .$$

9. Under the slow roll conditions, the inflationary equations become

$$H^2 \approx \frac{1}{3M_{\rm pl}^2} V(\phi) \quad \text{and} \quad 3H\dot{\phi} \approx -\frac{\partial V}{\partial \phi}\,.$$

Solve these equations to find the scale factor $a(t)$ for the choice of potentials

$$\text{i)} \qquad V(\phi) = \frac{1}{2} m^2 \phi^2\,;$$

$$\text{ii)} \qquad V(\phi) = \frac{1}{4} \lambda \phi^4\,.$$