Examples Sheet 4

1. A simple model of the spreading of an animal population $N(x,t)$ in a spatial domain is given by the nonlinear reaction-diffusion equation

$$N_t = D\,(N\,N_x)_x + \alpha N, \qquad N(x,0) = N_0\,\delta(x), \qquad N \to 0 \quad \text{as} \quad |x| \to \infty,$$

where D and N_0 are positive constants and α is a constant which may be positive or negative. By setting $N(x,t) = R(x,\tau)\,e^{\alpha t}$, where $\tau(t)$ is some time-like variable satisfying $\tau(0) = 0$, show that a suitable choice of τ yields $\quad R_\tau = (RR_x)_x, \quad R(x,0) = N_0\,\delta(x)$.

By setting $R(x,\tau) = \tau^{-1/3}F(\xi)$, $\xi = x/\tau^{1/3}$, show that the population is confined to a region $|x| < x_0$ where

$$x_0^3 = \frac{9N_0\,D}{2}\left(\frac{e^{\alpha t}-1}{\alpha}\right).$$

Describe the evolution of the population in the cases $\alpha = 0$, $\alpha > 0$ and $\alpha < 0$.

2. A bistable system with diffusion is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u(u-r)(u-1)$$

where $0 < r < 1$. Seek a travelling wave solution by setting $\xi = x - ct$ and $u(x,t) = f(\xi)$, and find the differential equation satisfied by f.

   (i) [*Approach 1 from lectures*] Rewrite your differential equation as two first order equations. Suppose that c takes the exact value that allows a travelling wave solution (you will need to consider $r < 1/2$ and $r > 1/2$ separately), sketch the phase plane for the system, marking the trajectory that corresponds to the travelling wave.

   (ii) [*Approach not seen in lectures*] Return to the differential equation for f and try this further approach: impose that the solution satisfies $f' = af(f-1)$ and find suitable a and c that yield a valid solution. By solving this first-order equation for f, give the corresponding solution for $u(x,t)$.

3. The SIR epidemic model can be extended to be a spatial model for the spread of an infectious disease:

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\beta IS \qquad + D\frac{\partial^2 S}{\partial x^2}\\ \frac{\partial I}{\partial t} &= +\beta IS - \nu I + D\frac{\partial^2 I}{\partial x^2}\,.\end{aligned}$$

Suppose that an epidemic wave arrives in a previously uninfected region (so $S \approx N$ and $I \approx 0$). Consider the dynamics near this wave front by taking

$$\begin{aligned} S &= N - u(\xi)\\ I &= v(\xi)\end{aligned}$$

with $\xi = x - ct$ and linearise in u and v. You may assume that the system will settle to the slowest possible wave speed. Find the wave speed of the epidemic, and show that it is proportional to $\sqrt{R_0 - 1}$.

4. Consider the chemotactic system

$$\begin{aligned}
\frac{\partial n}{\partial t} &= b\,n\left(1-\frac{n}{n_0}\right) - \frac{\partial}{\partial x}\left(n\,\chi(a)\frac{\partial a}{\partial x}\right) + D\frac{\partial^2 n}{\partial x^2} \\
\frac{\partial a}{\partial t} &= h\,n - d\,a + D_A\frac{\partial^2 a}{\partial x^2},
\end{aligned}$$

where

$$\chi(a) = \frac{\chi_0\,a_0}{(a_0+a)^2}\,.$$

Find a rescaling such that this reduces to

$$\begin{aligned}
\frac{\partial u}{\partial \tau} &= u(1-u) - \beta\frac{\partial}{\partial \xi}\left[\frac{u}{(\alpha+v)^2}\frac{\partial v}{\partial \xi}\right] + \frac{\partial^2 u}{\partial \xi^2} \\
\frac{\partial v}{\partial \tau} &= \gamma(u-v) + \delta\,\frac{\partial^2 v}{\partial \xi^2}\,.
\end{aligned}$$

[Hint: do the rescaling over a few steps, keeping an eye on the intended final form.]

Show that the uniform, steady solution $u = v = 1$ is unstable to a spatial perturbation if

$$\frac{\beta\gamma}{(1+\alpha)^2} > (\sqrt{\gamma}+\sqrt{\delta})^2.$$

Find the critical wavenumber in the case when $\alpha = \gamma = \delta = 1$.

5. Investigate the possibility of Turing instability for the reaction-diffusion system.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{u^2}{v} - b\,u + \nabla^2 u \\
\frac{\partial v}{\partial t} &= u^2 - v + d\,\nabla^2 v\,.
\end{aligned}$$

In particular, find the region of the parameter space (b, d) in which Turing instability can occur, and give the value for the critical wavenumber at the onset of instability in terms of d.

6. A space-dependent phytoplankton and zooplankton model can be reduced to the following equations

$$\begin{aligned}
\frac{\partial u}{\partial t} &= u + u^2 - \gamma\,u\,v + \nabla^2 u \\
\frac{\partial v}{\partial t} &= \beta\,u\,v - v^2 + d\,\nabla^2 v\,.
\end{aligned}$$

Find the regions in the $\beta - \gamma$ plane (a) in which there is a stable, homogeneous state (u_0, v_0) in which neither u_0 nor v_0 is zero and (b) in which that state may be unstable to a Turing instability. In case (b), for what values of d will the instability occur? Find the critical wavenumber for the onset of the instability in terms of β and d.

In addition to the examples sheets, students are encouraged to do the exercises given in lectures (solutions available on Moodle).