

1 Show that:

$$(i) \int_0^\infty \frac{dx}{(x^2 + 1)^2(x^2 + 4)} = \frac{\pi}{18};$$

$$(ii) \int_{-\infty}^\infty \frac{\cos x \, dx}{x^2 + a^2} = \frac{\pi}{a} e^{-a} \quad \text{where } a > 0;$$

$$(iii) \int_{-\infty}^\infty \frac{x - \sin x}{x^3} \, dx = \frac{\pi}{2};$$

2 Show that

$$(i) \int_0^\infty \frac{x^\beta \, dx}{1+x}, \quad \text{where } \beta \in \mathbb{C}, \text{ converges for } -1 < \operatorname{Re} \beta < 0.$$

$$(ii) \int_\gamma (1 + \tanh z) \, dz, \quad \text{where } \{\gamma : z = se^{i\alpha}, 0 \leq s < \infty\} \text{ and } \alpha \in \mathbb{R}, \text{ converges for } \alpha \in (-\pi, -\pi/2) \cup (\pi/2, \pi). \text{ Does the integral converge at } \alpha = \pi?$$

3 Let $f(t)$ be analytic at $t = 0$ with $f(0) = 0$ and $f'(0) \neq 0$. Let C be a circle centred on the origin, with interior D , such that f is analytic in D and the inverse of f exists on $f(D)$.

For a fixed point z within C , let $w = f(z)$. Assuming that w is small, show (using the residue theorem) that

$$z = \frac{1}{2\pi i} \int_C \frac{tf'(t)}{f(t) - w} dt,$$

and hence that $z = \sum_{n=1}^\infty b_n w^n$, where

$$b_n = \frac{1}{2\pi i} \int_C \frac{tf'(t)}{(f(t))^{n+1}} dt = \frac{1}{2\pi i n} \int_C \frac{1}{(f(t))^n} dt = \frac{1}{n!} \lim_{t \rightarrow 0} \frac{d^{n-1}}{dt^{n-1}} \left(\frac{t}{f(t)} \right)^n.$$

Show that the equation $w = ze^{-z}$ has a solution, for sufficiently small w (how small?),

$$z = \sum_{n=1}^\infty \frac{n^{n-1}}{n!} w^n.$$

Find also one solution of the equation $w = 2z - z^2$.

4 Let $\phi(x, y)$ be a harmonic function. Show that ϕ is the real part of any analytic function $f(z)$ of the form

$$f(z) = 2\phi((z+1)/2, (z-1)/2i) - \phi(1, 0) + ic$$

where c is a real constant (provided ϕ is such that the right hand side exists). Use this formula to find analytic functions whose real parts are (i) $x/(x^2 + y^2)$ and (ii) $\tan^{-1} y/x$.

[Note: You might like to start by considering a harmonic conjugate $\psi(x, y)$, where ϕ and ψ obey the Cauchy-Riemann conditions, and then write $f(z) = \phi(x, y) + i\psi(x, y) = \sum_{n=0}^\infty a_n(z-1)^n$.]

5 Let $P(z)$ be a polynomial of degree n , with n roots, none of which lie on a simple closed contour L . Show that

$$\frac{1}{2\pi i} \int_L \frac{P'(z)}{P(z)} dz = \text{number of roots lying within } L,$$

where the roots should be counted according to their multiplicity.

[Note: Try to do this question without assuming *the fundamental theorem of algebra*.]

6 Consider a rectangular contour C , with corners at $(N + \frac{1}{2})(\pm 1 \pm i)$ to evaluate

$$\frac{1}{2\pi i} \int_C \frac{\pi \cot \pi z \coth \pi z}{z^3} dz.$$

In the limit as $N \rightarrow \infty$, show that

$$\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^3} = \frac{7}{180} \pi^3.$$

7 Evaluate the following integrals, where $|f(z)/z| \rightarrow 0$ as $|z| \rightarrow \infty$ and $f(z)$ is analytic in the upper half plane (including the real axis):

$$(i) \mathcal{P} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx \quad (ii) \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x(x-i)} dx \quad (iii) \mathcal{P} \int_{-\infty}^{\infty} \frac{dx}{x-i} \quad (iv) \mathcal{P} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x} dx.$$

Which of the integrals (i), (iii) and (iv) is real?

8 Use the principal value technique to evaluate

$$\int_0^{\infty} \frac{\sin x}{x(x^2+1)} dx.$$

9 By considering the function

$$\frac{e^{iz} - 1}{z},$$

compute the Hilbert transforms of

$$\frac{\cos x - 1}{x} \quad \text{and} \quad \frac{\sin x}{x}.$$

10 Let $f_1(z)$ be the branch of $(z^2 - 1)^{\frac{1}{2}}$ defined by branch cuts in the z -plane along the real axis from -1 to $-\infty$ and from 1 to ∞ , with $f_1(z)$ real and positive just above the latter cut. Let $f_2(z)$ be the branch of $(z^2 - 1)^{\frac{1}{2}}$ defined by a cut along the real axis from -1 to $+1$, with $f_2(x)$ real and positive for $(x - 1)$ real and positive. Show that $f_1(z) = f_1(-z)$ but $f_2(z) = -f_2(-z)$.

11 By integrating the function

$$\frac{(\ln z)^2}{z^2 + 1}$$

around an appropriate contour, compute the following integrals:

$$\int_0^\infty \frac{(\ln x)^m}{x^2 + 1} dx, \quad m = 1, 2.$$

12 Evaluate

$$\int_0^\infty \frac{x^{m-1}}{x^2 + 1} dx, \quad 0 < m < 2.$$

Why is it necessary for m to satisfy the above restrictions?