Mathematical Tripos Part II - C Course — Lent term 2021

Further Complex Methods, Examples sheet 2 — Dr Berry Groisman

Comments and corrections: e-mail to bg268@cam.ac.uk.

The starred parts of questions are intended as extras: attempt them if you have time, but not at the expense of unstarred questions.

1 Let

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{uz}}{1+e^u}\, du\ .$$

For what region of the z-plane does $F(z)$ define an analytic function?

Show by closing the contour (use a rectangle) in the upper half plane that

$$F(z) = \pi \operatorname{cosec} \pi z\ .$$

Explain how this result provides the analytic continuation of $F(z)$.

2 Define the branch of $f(z) = (1-z^2)^{\frac{1}{2}}$ by the branch cut along the real axis from -1 to $-\infty$ and from 1 to ∞, with $f(0) = 1$. Use this branch and a suitably chosen semi-circular contour (with finite radius R greater than 1) in the upper half plane to evaluate

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}}\, dx\ .$$

3 The function $\sin^{-1} z$ is defined, for $0 \le \arg z < 2\pi$, by

$$\sin^{-1} z = \int_0^z \frac{dt}{\sqrt{1-t^2}}\ ,$$

where the integrand has a branch cut along the real axis from -1 to $+1$ and takes the value $+1$ at the origin on the upper side of the cut. The path of integration is a straight line for $0 \le \arg(z) \le \pi$ and is curved in a positive sense round the branch cut for $\pi < \arg z < 2\pi$. Express $\sin^{-1}(e^{i\pi} z)$ $(0 < \arg z < \pi)$ in terms of $\sin^{-1} z$ and deduce that $\sin(\phi - \pi) = -\sin\phi$. *Hint:* $\sin^{-1}(e^{i\pi}z) = -\pi + \sin^{-1} z$, as can be derived by calculating the integral half way round the cut and remembering that the integrand is an odd function.

4 Let $\omega_{m,n} = m\omega_1 + n\omega_2$, where (m,n) are integers not both zero, and let

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{m,n}^{\infty} \left[\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{{\omega_{m,n}}^2} \right]$$

be the Weierstrass elliptic function with periods (ω_1, ω_2) such that ω_1/ω_2 is not real. Show that, in a neighbourhood of $z = 0$,

$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6)$$

where

$$g_2 = 60 \sum_{m,n} (\omega_{m,n})^{-4}, \quad g_3 = 140 \sum_{m,n} (\omega_{m,n})^{-6}.$$

Deduce that $\mathcal{P}$ satisfies a 1st order nonlinear ODE

$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2 \mathcal{P} - g_3.$$

5 Show that

$$4\mathcal{P}(2z) - \left(\frac{\mathcal{P}''(z)}{\mathcal{P}'(z)}\right)^2 + 8\mathcal{P}(z) = 0.$$

6 By using a contour consisting of the boundary of a quadrant, indented at the origin, show that (for a range of z to be stated)

$$\int_0^\infty t^{z-1}e^{-it}\,dt = e^{-\frac{1}{2}\pi iz}\Gamma(z).$$

Hence evaluate (again, for ranges of z to be stated)

$$\int_0^\infty t^{z-1}\cos t\,dt \quad \text{and} \quad \int_0^\infty t^{z-1}\sin t\,dt.$$

Use your results to evaluate $\displaystyle\int_0^\infty \frac{\cos t}{t^{1/2}}\,dt$, $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt$ and $\displaystyle\int_0^\infty \frac{\sin t}{t^{3/2}}\,dt$.

7 Derive the formula $\mathrm{B}(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta$ and prove that

$$\mathrm{B}(z,z) = 2^{1-2z}\mathrm{B}(z,\tfrac{1}{2}).$$

For which values of z does this result hold?

8 Show, using properties of the B function, that

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \tfrac{1}{\sqrt{32\pi}}\left(\Gamma(\tfrac{1}{4})\right)^2$$

Using the change of variable $x = t(2-t^2)^{-\frac{1}{2}}$, deduce that

$$\mathrm{K}(\tfrac{1}{\sqrt{2}}) = \tfrac{4}{\sqrt{\pi}}\left(\Gamma(\tfrac{5}{4})\right)^2 ,$$

where $\mathrm{K}(k)$ is the complete elliptic integral $\displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$.

9 Starting with the infinite product representation of the Gamma function (Weierstrass canonical product) and using the definition of γ, derive the Euler's product formula, i.e.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\;n^z}{z(1+z)(2+z)\dots(n+z)} .$$

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(a) Use Stirling's approximation $\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}/n! \to 1$ as $n \to \infty$ and the Euler's product formula to show that

$$\Gamma_n(z) := \frac{\sqrt{2\pi}e^{-n}n^{z+n+\frac{1}{2}}}{z(z+1)\cdots(z+n)} \to \Gamma(z)$$

as $n \to \infty$.

Hence, prove that

$$\frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)}$$

is a constant independent of z. Then, by letting $z \to \frac{1}{2}$ evaluate the relevant constant and thus establish the following identity:

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z).$$

(b) **Futhermore, by constructing $\Gamma_n(z)\Gamma_n(z+\frac{1}{m})\cdots\Gamma_n(z+\frac{m-1}{m})/\Gamma_{nm}(mz)$, prove the Gauss multiplication formula*

$$\Gamma(z)\Gamma(z+\frac{1}{m})\Gamma(z+\frac{2}{m})\ldots\Gamma(z+\frac{m-1}{m}) = (2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mz}\Gamma(mz),$$

for $m = 1, 2, \ldots$ and $mz \neq 0, -1, -2, \ldots$.

11 Using $t = s\tau$, $s > 0$, it follows that

$$\frac{\Gamma(z)}{s^z} = \int_0^\infty e^{-s\tau}\tau^{z-1}d\tau.$$

Letting $z = 1$ and integrating the resulting formula with respect to s from 1 to t, show that

$$\ln t = \int_0^\infty \left(e^{-\tau} - e^{-t\tau}\right)\frac{d\tau}{\tau}.$$

Using this formula in the expression for $\Gamma'(z)$, prove that

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left(e^{-\tau} - \frac{1}{(1+\tau)^z}\right)\frac{d\tau}{\tau}.$$

Hence, deduce that

$$\gamma = -\int_0^\infty \left(e^{-\tau} - \frac{1}{1+\tau}\right)\frac{d\tau}{\tau}\ .$$

12 (a) Prove that for $\text{Re} z > 1$,

$$\frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt = \frac{\Gamma(1-z)}{2i\pi} \int_\gamma \frac{t^{z-1}}{e^{-t} - 1}\, dt,$$

where γ denotes the Hankel contour. Hence, deduce that the RHS of the above equation provides the analytic continuation of Riemann's zeta function.

(b) The Bernoulli numbers B_n are defined by

$$\frac{1}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^{m-1}}{m!},$$

and $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_{2m+1} = 0$ for $m = 1, 2, \ldots$.

Use (a) and the residue theorem to compute $\zeta(-n)$, $n = 0, 1, 2, \ldots$ in terms of B_n. Hence, deduce that the negative even integers are zeros of $\zeta(z)$.