1 Let
\[ F(z) = \int_{-\infty}^{\infty} \frac{e^{uz}}{1 + e^u} du . \]

For what region of the \( z \)-plane does \( F(z) \) define an analytic function?

Show by closing the contour (use a rectangle) in the upper half plane that
\[ F(z) = \pi \csc \pi z . \]

Explain how this result provides the analytic continuation of \( F(z) \).

2 Let \( \omega_{m,n} = m\omega_1 + n\omega_2 \), where \((m, n)\) are integers not both zero, and let
\[ P(z) = \frac{1}{z^2} + \sum_{m,n} \left[ \frac{1}{(z - \omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2} \right] \]
be the Weierstrass elliptic function with periods \((\omega_1, \omega_2)\) such that \( \omega_1/\omega_2 \) is not real. Show that, in a neighbourhood of \( z = 0 \),
\[ P(z) = \frac{1}{z^2} + \frac{1}{20}g_2z^2 + \frac{1}{28}g_3z^4 + O(z^6) \]
where
\[ g_2 = 60\sum_{m,n}(\omega_{m,n})^{-4}, \quad g_3 = 140\sum_{m,n}(\omega_{m,n})^{-6}. \]

Deduce that \( P \) satisfies a 1st order nonlinear ODE
\[ (P')^2 = 4P^3 - g_2P - g_3. \]

3 Show that
\[ 4P(2z) - \left( \frac{P''(z)}{P'(z)} \right)^2 + 8P(z) = 0. \]

4 The function \( \sin^{-1}z \) is defined, for \( 0 \leq \arg z < 2\pi \), by
\[ \sin^{-1}z = \int_{0}^{z} \frac{dt}{\sqrt{1-t^2}} , \]
where the integrand has a branch cut along the real axis from \(-1\) to \(+1\) and takes the value \(+1\) at the origin on the upper side of the cut. The path of integration is a straight line for \( 0 \leq \arg(z) \leq \pi \) and is curved in a positive sense round the branch cut for \( \pi < \arg z < 2\pi \).

Express \( \sin^{-1}(e^{i\pi}z) \) \((0 < \arg z < \pi)\) in terms of \( \sin^{-1}z \) and deduce that \( \sin(\phi - \pi) = -\sin \phi \).

**Hint:** \( \sin^{-1}(e^{i\pi}z) = -\pi + \sin^{-1}z \), as can be derived by calculating the integral half way round the cut and remembering that the integrand is an odd function.
Find two independent solutions of the Airy equation \( w'' - zw = 0 \) in the form
\[
w(z) = \int_{\gamma} e^{zt} f(t) \, dt,
\]
where \( \gamma \) is to be specified in each case. Show that there is a solution for which \( \gamma \) can be chosen to consist of two straight line segments in the left half \( t \)-plane (\( \text{Re} \, t \leq 0 \)).

For this solution show that, if \( w'(0) = -iA \frac{3}{6} \Gamma(1/3) \), where \( A \) is a constant, then \( \frac{w'(0)}{A} = 1 \frac{3}{6} \Gamma(2/3) \).

By writing \( w(z) \) in the form of an integral representation with the Laplace kernel show that the confluent hypergeometric equation \( zw'' + (c - z)w' - aw = 0 \) has solutions of the form
\[
w(z) = \int_{\gamma} t^{a-1}(1-t)^{c-a-1} e^{t \gamma} \, dt,
\]
provided the path \( \gamma \) is chosen such that \( \left[ t^{a-1}(1-t)^{c-a-1} e^{t \gamma} \right]_\gamma = 0 \).

In the case Re \( z > 0 \), find paths which provide two independent solutions in each of the following cases (where \( m \) is a positive integer):
(i) \( a = -m \), \( c = 0 \);
(ii) \( \text{Re} \, a < 0 \), \( c = 0 \), \( a \) is not an integer;
(iii) \( a = 0 \), \( c = m \);
(iv) \( \text{Re} \, c > \text{Re} \, a > 0 \), \( c \) and \( c - a \) are not integers.

Use the Laplace transform to solve the ordinary differential equation
\[
\frac{d^2 y}{dt^2} - k^2 y = f(t), \quad k > 0, \quad y(0) = y_0, \quad y'(0) = y'_0.
\]

Let \( f(t) = e^{-k_0 t} \), \( k_0 \neq k \), \( k_0 > 0 \), so that the Laplace transform of \( f(t) \) is
\[
\hat{f}(s) = \frac{1}{s + k_0}.
\]

Show that
\[
y(t) = y_0 \cosh kt + \frac{y'_0}{k} \sinh kt + \frac{e^{-k_0 t}}{k_0^2 - k^2} \cosh kt - \frac{e^{-k_0 t}}{k_0^2 - k^2} \sinh kt + \frac{k_0}{k} \frac{k_0}{k_0^2 - k^2} \sinh kt.
\]

Now suppose that \( f(t) \) is an arbitrary continuous function that possesses a Laplace transform. Use the convolution theorem for Laplace transforms, or otherwise, to show that
\[
y(t) = y_0 \cosh kt + \frac{y'_0}{k} \sinh kt + \int_0^t f(t') \sinh k(t - t') \frac{k}{k_0} dt'.
\]

Put \( f(t) = e^{-k_0 t} \) and re-obtain your answer to the first part of this question. Suppose now that \( k_0 = k \). What is \( y(t) \)? Could you have found this solution by taking the limit in (1) as \( k_0 \to k \)?
The Schrödinger equation is
\[ i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0. \]

Suppose that \( u(x, 0) = f(x) \).
Fourier transform this equation with respect to \( x \) to find
\[ u(x, t) = e^{-i\pi/4} \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{i(x - x')^2}{4t}} f(x') dx'. \]
(You may it useful to recall that \( \int_{-\infty}^{\infty} e^{iu^2} du = e^{i\pi/4} \sqrt{\pi} \).

Now use Laplace transform methods to find the same solution to this problem.