Further Complex Methods, Examples sheet 3 | Dr Berry Groisman

Comments and corrections: e-mail to bg268@cam.ac.uk.

1 Show that

$$E_1(k) = \int_k^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \ln k + k - \frac{k^2}{4} + O(k^3), \quad k \to 0^+.$$

Hint:

$$E_1(k) = \int_k^\infty \frac{dt}{t(t+1)} + \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t} - \int_0^k \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t}.$$

2

Show that for $Re\, z > 1$

$$(1 - 2^{1-z})\zeta(z) = (1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} \cdots) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t + 1}dt.$$

[Note: This result is actually valid for $Re\, z > 0$.]

3

Show that

$$\int_{-\infty}^{(0+)} \frac{\ln t}{e^{-t} - 1}dt = 0.$$

Hence show that

$$\lim_{z\to 1}(\zeta(z) - (z-1)^{-1}) = \gamma,$$

and

$$\zeta'(0) = -\ln\sqrt{2\pi}.$$

4

The psi-function is defined to be

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z).$$

Show that

$$\psi'(z) = \sum_{s=0}^\infty \frac{1}{(s+z)^2}, \qquad (z \neq 0, -1, -2\cdots).$$

Then show that when z is real and positive, that $\Gamma(z)$ has a single minimum which lies between $z = 1$ and $z = 2$.

Show that

$$\ln\Gamma(z) = -\gamma(z-1) + \sum_{s=2}^\infty (-1)^s\frac{\zeta(s)}{s}(z-1)^s.$$

5 Find two independent solutions of the Airy equation $w'' - zw = 0$ in the form

$$w(z) = \int_\gamma e^{zt} f(t)\, dt,$$

where γ is to be specified in each case. Show that there is a solution for which γ can be chosen to consist of two straight line segments in the left half t-plane (Re $t \le 0$).

For this solution show that, if $w(z)$ is normalised so that $w(0) = iA\, 3^{-\frac{1}{6}}\Gamma(1/3)$, where A is a constant, then $w'(0) = -iA\, 3^{\frac{1}{6}}\Gamma(2/3)$.

[Note: $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ for $Re\, z > 0$.]

6 By writing $w(z)$ in the form of an integral representation with the Laplace kernel show that the confluent hypergeometric equation $zw'' + (c - z)w' - aw = 0$ has solutions of the form

$$w(z) = \int_\gamma t^{a-1}(1-t)^{c-a-1}e^{tz}\, dt,$$

provided the path γ is chosen such that $[t^a(1-t)^{c-a}e^{tz}]_\gamma = 0$.

In the case $\mathrm{Re}\, z > 0$, find paths which provide two independent solutions in each of the following cases (where m is a positive integer):

(i) $a = -m$, $c = 0$;
(ii) $\mathrm{Re}\, a < 0$, $c = 0$, a is not an integer;
(iii) $a = 0$, $c = m$;
(iv) $\mathrm{Re}\, c > \mathrm{Re}\, a > 0$, a and $c - a$ are not integers.

7 Use the Laplace transform to solve the ordinary differential equation

$$\frac{d^2y}{dt^2} - k^2 y = f(t), \quad k > 0, \quad y(0) = y_0, \quad y'(0) = y_0'.$$

Let $f(t) = e^{-k_0 t}, k_0 \neq k, k_0 > 0$, so that the Laplace transform of $f(t)$ is

$$\hat{f}(s) = \frac{1}{s + k_0}.$$

Show that

$$y(t) = y_0 \cosh kt + \frac{y_0'}{k} \sinh kt + \frac{e^{-k_0 t}}{k_0^2 - k^2} - \frac{\cosh kt}{k_0^2 - k^2} + \frac{\frac{k_0}{k}}{k_0^2 - k^2} \sinh kt. \qquad (1)$$

Now suppose that $f(t)$ is an arbitrary continuous function that possesses a Laplace transform. Use the convolution theorem for Laplace transforms, or otherwise, to show that

$$y(t) = y_0 \cosh kt + \frac{y_0'}{k} \sinh kt + \int_0^t f(t') \frac{\sinh k(t - t')}{k} dt'.$$

Put $f(t) = e^{-k_0 t}$ and re-obtain your answer to the first part of this question. Suppose now that $k_0 = k$. What is $y(t)$? Could you have found this solution by taking the limit in (1) as $k_0 \to k$?

8

The Schrödinger equation is

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0.$$

Suppose that $u(x, 0) = f(x)$.
Fourier transform this equation with respect to x to find

$$u(x,t) = \frac{e^{-i\pi/4}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{i(x-x')^2}{4t}} f(x')dx'.$$

(You may it useful to recall that $\int_{-\infty}^{\infty} e^{iu^2} du = e^{\frac{i\pi}{4}}\sqrt{\pi}$.)

Now use Laplace transform methods to find the same solution to this problem.