

*Comments and corrections: e-mail to bg268@cam.ac.uk.*

**1** Show that the most general linear second order ordinary differential equation whose only singularities are regular singular points at  $z = a$  and  $z = b$  can be written in the form

$$w'' + \left[ \frac{1-A}{z-a} + \frac{1+A}{z-b} \right] w' + \frac{B(a-b)^2}{(z-a)^2(z-b)^2} w = 0, \quad (\dagger)$$

where  $A$  and  $B$  are arbitrary constants.

Write down and solve the equation when the two singular points are at 0 and  $\infty$ , in the two cases  $A^2 \neq 4B$  and  $A^2 = 4B$ . Use a Möbius transformation to deduce the general solution of  $(\dagger)$ . What is the significance of the two constants  $\alpha$  and  $\alpha'$  which satisfy  $\alpha + \alpha' = A$  and  $\alpha\alpha' = B$ ?

**2** Consider the second order linear differential equation with three, and only three, regular singular points in the finite complex plane, all other points being ordinary points,

$$\frac{d^2y}{dz^2} + \left( \frac{A}{z-z_1} + \frac{B}{z-z_2} + \frac{C}{z-z_3} \right) \frac{dy}{dz} + \frac{1}{(z-z_1)(z-z_2)(z-z_3)} \left( \frac{D}{z-z_1} + \frac{E}{z-z_2} + \frac{F}{z-z_3} \right) y = 0.$$

Show that by making a fractional linear transformation

$$z \rightarrow z' = \frac{(z_3 - z_2)(z - z_1)}{(z_3 - z_1)(z - z_2)}$$

that the new differential equation is of the same form but now with regular singular points at 0, 1,  $\infty$  with the same roots of the indicial equation as in the original equation.

Can a similar transformation be made to map 0, 1,  $\infty$  into any three points in the finite complex plane?

**3**

Consider the equation for  $y(z)$  with  $P$ -symbol

$$P \left\{ \begin{matrix} 0 & 1 & \infty \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix} \right\} z.$$

By transforming  $y \rightarrow \tilde{y}(z) = z^{-\alpha_1}(1-z)^{-\beta_1}y(z)$ , show that the resulting equation for  $\tilde{y}(z)$  is the hypergeometric equation.

4 By expanding  $(1 - tz)^{-a}$ , show that

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(c-b)}{\Gamma(a)} \sum_0^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!},$$

where  $(1 - tz)^{-a}$  takes its principal value, provided  $\operatorname{Re} c > \operatorname{Re} b > 0$ , and  $|z| < 1$ . You should explain the reason for these conditions.

State the regions of the complex  $z$ -plane in which (i) the integral defines an analytic function and (ii) the sum define an analytic function. Explain how the integral provides an analytic continuation in  $z$  of the function defined by the sum.

Given that the above integral is a multiple of  $F(a, b; c; z)$ , show that

$$F(a, b; c; 1) = \frac{\Gamma(c-b-a)\Gamma(c)}{\Gamma(c-b)\Gamma(c-a)}.$$

when  $a, b$  and  $c$  satisfy a condition that you should give.

5 What can be said about the nature of the solutions of a second order linear ordinary differential equation in the neighbourhood of a regular singular point?

Explain carefully why the hypergeometric equation with the usual parameters  $a, b$ , and  $c$  (so that the exponents at  $z = 0$  are 0 and  $1 - c$ ), has, for any given value of the parameter  $c$ , a solution  $w(z)$  that satisfies  $w(0) = 1$ . Is  $w(z)$  uniquely determined?

Is it the case that, for any given value of the parameter  $c$ , there is a solution  $w(z)$  that is analytic at  $z = 0$ , and satisfies  $w(0) = 1$ ? If such a solution exists, is it unique?

6 Is  $(1 - z)^{c-a-b}F(c-a, c-b; c; z)$  analytic at  $z = 0$ ? The branch of  $(1 - z)^{c-a-b}$  is defined by  $|\arg(1 - z)| < \pi$  (which implies a branch cut from 1 to  $\infty$  along the positive real axis). [Assume  $c \neq 0, -1, -2, \dots$ ]

Show, by considering transformations of  $P$ -functions, that

$$(1 - z)^{c-a-b}F(c-a, c-b; c; z) = F(a, b; c; z).$$

Show also that

$$(1 - z)^{-a}F(a, c-b; c; \frac{z}{z-1}) = F(a, b; c; z).$$

**7** The hypergeometric function,  $F(a, b; c; z)$  may be defined for  $|z| < 1$  (and as usual  $c \neq 0, -1, -2, \dots$ ) by

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

Let

$$g(z) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)} \Gamma(-t)(-z)^t dt$$

where the contour runs to the left of all poles of  $\Gamma(-t)$  and to the right of all poles of  $\Gamma(a+t)$  and  $\Gamma(b+t)$ ,  $|\arg(-z)| < \pi$  and  $a \neq 0, -1, -2, \dots$ ,  $b \neq 0, -1, -2, \dots$

(i) Give a sketch of the complex  $t$ -plane showing the positions of the singularities of the integrand and the curve.

By closing the contour with a large semi-circle in the right half complex plane (which may be assumed to make a negligible contribution to the integral), show that  $g(z) = F(a, b; c; z)$ .

(ii) By closing the contour instead with a large semi-circle in the left half complex plane (which may be assumed to make a negligible contribution to the integral), show that the analytic continuation to  $|z| > 1$  of the series for  $F(a, b; c; z)$  in the case when  $a$  and  $b$  do not differ by an integer is provided by

$$\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)} (-z)^{-a} F(a, 1-c+a; 1-b+a; z^{-1}) + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)} (-z)^{-b} F(b, 1-c+b; 1-a+b; z^{-1})$$

**8** By changing variable in the hypergeometric equation, deduce that

$$F(a, b; 1+a+b-c; 1-z) \equiv y_1(z)$$

is a solution of the hypergeometric equation near  $z = 1$  and deduce that another solution is

$$(1-z)^{c-a-b} F(c-a, c-b; 1+c-a-b; 1-z) \equiv y_2(z).$$

[Assume  $c \neq 0, -1, -2, \dots$ ]

Use the integral representation

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt,$$

which is valid if  $0 < \arg(z-1) < 2\pi$  and  $\operatorname{Re} c > \operatorname{Re} b > 0$ , to show that

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}, \quad (\dagger\dagger)$$

provided the above conditions hold, and  $\operatorname{Re}(c-a-b) > 0$ .

Find constants  $A$  and  $B$  such that (under conditions to be specified)

$$F(a, b; c; z) = Ay_1(z) + By_2(z).$$