

Example Sheet 3

1. Inertia tensor

- (a) Prove that the principal moments of inertia, (I_1, I_2, I_3) , are real and non-negative.
- (b) During the lectures we outlined the proof of the Parallel Axis Theorem, which is a statement that the inertia tensor about a point P , which is displaced by \mathbf{c} from the centre of mass C , is related to the inertia tensor about C by

$$I_{ab}^P = I_{ab}^C + M(c^2\delta_{ab} - c_a c_b),$$

where M is the total mass of the body. Complete the proof of the theorem. (It will be helpful to choose the origin to be at the centre of mass.)

2. Euler angles

The rotation matrix that relates the body axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to the space axes $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is given in terms of the Euler angles $(\theta(t), \phi(t), \psi(t))$ by

$$\begin{aligned} \mathbf{R} &= \mathbf{R}_3(\psi)\mathbf{R}_n(\theta)\mathbf{R}_z(\phi) \\ &= \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

- (a) Calculate the matrix $\mathbf{A} = \dot{\mathbf{R}}\mathbf{R}^\top$ and verify that it is antisymmetric. By identifying \mathbf{A} as the antisymmetric matrix associated with the angular velocity vector in the body frame, deduce that

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi \\ -\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi \\ \dot{\psi} + \dot{\phi} \cos \theta \end{pmatrix}.$$

- (b) * [optional extra] Calculate the matrix $\mathbf{B} = \mathbf{R}^\top \dot{\mathbf{R}}$ and verify that it is antisymmetric. By identifying \mathbf{B} as the antisymmetric matrix associated with the angular velocity vector in the space frame, deduce that

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi \\ \dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi \\ \dot{\phi} + \dot{\psi} \cos \theta \end{pmatrix}.$$

3. Free symmetric top

- (a) Consider the torque-free motion of a round plate. Show that, in the body frame, the angular velocity vector $\boldsymbol{\omega}$ precesses around the body axis \mathbf{e}_3 perpendicular to the plate with constant angular frequency equal to ω_3 .
- (b) The physicist Richard Feynman tells the following story:

“I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.

I had nothing to do, so I start figuring out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate – two to one. It came out of a complicated equation! I went on to work out equations for wobbles. Then I thought about how the electron orbits start to move in relativity. Then there’s the Dirac equation in electrodynamics. And then quantum electrodynamics. And before I knew it...the whole business that I got the Nobel prize for came from that piddling around with the wobbling plate.”

[Here the ‘wobble’ is associated with precession of the top about \mathbf{e}_z (motion in ϕ), not nutation (motion in θ).]

Feynman was right about quantum electrodynamics. But what about the plate?

[You could try two alternative methods. First, by using the expression for ω_3 in terms of Euler’s angles together with the result of part (a). Second, by writing down the Lagrangian of the top and deriving the equation of motion for θ .]

- (c) Consider a uniform symmetric ellipsoid of mass M with semi-axes $a = b \neq c$ (see Example 2.5(e)). Find the ratio of the semi-axes for which $\dot{\phi}$, the angular frequency of precession of the top about the angular momentum \mathbf{L} , equals $\omega_3/(5 \cos \theta)$. Deduce further that $\dot{\psi} = \frac{4}{5}\omega_3$. What is the relationship between $\dot{\phi}$ and $\dot{\psi}$ for small values of θ ? Compare with the result obtained in part (b).

4. Free asymmetric top (1)

- (a) Throw a book in the air. (Secure it with an elastic band first!) If the principal moments of inertia are $I_3 > I_2 > I_1$, convince yourself that the book can rotate in a stable manner about the principal axes \mathbf{e}_1 and \mathbf{e}_3 , but not about \mathbf{e}_2 .
- (b) Use Euler’s equations to show that the energy E and the squared angular momentum $|\mathbf{L}|^2$ of a free asymmetric top are conserved. Suppose that the initial conditions are such that

$$|\mathbf{L}|^2 = 2EI_2,$$

with the initial angular velocity $\boldsymbol{\omega}$ perpendicular to the intermediate principal axis \mathbf{e}_2 . Show that $\boldsymbol{\omega}$ will ultimately end up parallel to \mathbf{e}_2 . What is the characteristic timescale required to reach this steady state?

5. *Free asymmetric top (2)*

A rigid lamina (i.e. a two-dimensional object) has principal moments of inertia about the centre of mass given by

$$I_1 = (\mu^2 - 1), \quad I_2 = (\mu^2 + 1), \quad I_3 = 2\mu^2,$$

where $\mu > 1$. Write down Euler's equations for the lamina moving freely in space. Show that the component of the angular velocity in the plane of the lamina (i.e. $\sqrt{\omega_1^2 + \omega_2^2}$) is constant in time.

Choose the initial angular velocity to be $\boldsymbol{\omega} = \mu N \mathbf{e}_1 + N \mathbf{e}_3$. Define $\tan \alpha = \omega_2 / \omega_1$, which is the angle the component of $\boldsymbol{\omega}$ in the plane of the lamina makes with \mathbf{e}_1 . Show that it satisfies

$$\ddot{\alpha} + N^2 \cos \alpha \sin \alpha = 0$$

and deduce that, at time t ,

$$\boldsymbol{\omega} = \mu N \operatorname{sech}(Nt) \mathbf{e}_1 + \mu N \tanh(Nt) \mathbf{e}_2 + N \operatorname{sech}(Nt) \mathbf{e}_3.$$

6. *Lagrange top*

Consider a heavy symmetric top of mass M , fixed at the point P which is a distance l from the centre of mass. The principal moments of inertia about P are (I_1, I_1, I_3) and the Euler angles are defined as in the lectures. The top is spun with initial conditions $\dot{\phi} = 0$ and $\theta = \theta_0$. Show that θ obeys the equation of motion

$$I_1 \ddot{\theta} = -\frac{dV_{\text{eff}}}{d\theta},$$

where

$$V_{\text{eff}}(\theta) = \frac{I_3^2 \omega_3^2 (\cos \theta - \cos \theta_0)^2}{2I_1 \sin^2 \theta} + Mgl \cos \theta.$$

Suppose that the top is spinning very fast so that

$$I_3 \omega_3 \gg \sqrt{MglI_1}.$$

Show that the minimum of $V_{\text{eff}}(\theta)$ is close to θ_0 . Use this fact to deduce that the top nutates with angular frequency

$$\Omega \approx \frac{I_3}{I_1} \omega_3,$$

and sketch the subsequent motion.

7. *Lagrange top in Hamiltonian formalism*

The Lagrangian for the heavy symmetric top is

$$L = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi} \cos \theta)^2 - Mgl \cos \theta.$$

Find the conjugate momenta p_θ , p_ϕ and p_ψ and the Hamiltonian $H(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi)$. Derive Hamilton's equations.

8. *Hamilton's equations*

A system with two degrees of freedom x and y has the Lagrangian

$$L = xy + y\dot{x}^2 + \dot{x}\dot{y}.$$

Derive Lagrange's equations. Obtain the Hamiltonian $H(x, y, p_x, p_y)$. Derive Hamilton's equations and show that they are equivalent to Lagrange's equations.

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