

### Example Sheet 4

1. Verify the Jacobi identity for Poisson brackets,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

2. A particle with mass  $m$ , position  $\mathbf{r}$  and momentum  $\mathbf{p}$  has angular momentum  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ . Evaluate  $\{x_i, L_j\}$ ,  $\{p_i, L_j\}$ ,  $\{L_i, L_j\}$  and  $\{L_i, |\mathbf{L}|^2\}$ .

The Laplace–Runge–Lenz vector is defined as

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk \hat{\mathbf{r}},$$

where  $k$  is a constant and  $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ . Show that  $\{L_i, A_j\} = \epsilon_{ijk} A_k$ . For a system described by the Hamiltonian

$$H = \frac{|\mathbf{p}|^2}{2m} - \frac{k}{|\mathbf{r}|},$$

show, using Poisson brackets, that  $\mathbf{A}$  is conserved.

3. A particle of charge  $q$  moves in a time-independent background magnetic field  $\mathbf{B}$ . Show that  $\{m\dot{x}_i, m\dot{x}_j\} = q\epsilon_{ijk} B_k$  and  $\{x_i, m\dot{x}_j\} = \delta_{ij}$ .

A *magnetic monopole* is a particle that produces a radial magnetic field of the form

$$\mathbf{B} = g \frac{\hat{\mathbf{r}}}{r^2},$$

where  $g$  is a constant and  $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ . Consider a charged particle moving in the background of the magnetic monopole. Define the generalized angular momentum,

$$\mathbf{J} = m \mathbf{r} \times \dot{\mathbf{r}} - qg \hat{\mathbf{r}}.$$

Show that  $\{\mathbf{J}, H\} = \mathbf{0}$ . Why does this imply that  $\mathbf{J}$  is conserved?

4. In the lectures we constructed canonical transformations using generating functions. Consider canonical transformations  $\mathbf{q} \mapsto \mathbf{Q}(\mathbf{q}, \mathbf{p})$ ,  $\mathbf{p} \mapsto \mathbf{P}(\mathbf{q}, \mathbf{p})$  from the following perspective. Define the  $2n$ -dimensional vector  $\mathbf{x} = (q_1, \dots, q_n, p_1, \dots, p_n)^\top$  and the  $2n \times 2n$  matrix

$$\Omega = \begin{pmatrix} 0 & \mathbf{I}_n \\ -\mathbf{I}_n & 0 \end{pmatrix},$$

where each entry is itself an  $n \times n$  matrix.

- (a) Write Hamilton's equations for  $\dot{\mathbf{x}}$  in terms of  $\Omega$  and the Hamiltonian  $H$ .

(b) Hence deduce the following equation for the vector  $\mathbf{X} = (Q_1, \dots, Q_n, P_1, \dots, P_n)^\top$ :

$$\dot{\mathbf{X}} = (\mathbf{J}\Omega\mathbf{J}^\top) \frac{\partial H}{\partial \mathbf{X}},$$

where  $J^A_B = \partial X^A / \partial x^B$  ( $A, B = 1, \dots, 2n$ ) is the Jacobian matrix of the transformation. This implies that, if the Jacobian of a transformation satisfies

$$\mathbf{J}\Omega\mathbf{J}^\top = \Omega,$$

then Hamilton's equations are invariant under that transformation. The transformations with such a Jacobian (said to be *symplectic*) are canonical.

(c) Use the above conclusion to prove that, if the Poisson bracket structure is preserved, then the transformation is canonical.

5. Show that the following transformations are canonical:

$$(a) \quad P = \frac{1}{2}(p^2 + q^2), \quad Q = \arctan\left(\frac{q}{p}\right),$$

$$(b) \quad P = \frac{1}{q}, \quad Q = pq^2,$$

$$(c) \quad P = 2\sqrt{q}(1 + \sqrt{q} \cos p) \sin p, \quad Q = \log(1 + \sqrt{q} \cos p).$$

6. Show that the following transformation is canonical, for any constant  $\lambda$ :

$$\begin{aligned} q_1 &= Q_1 \cos \lambda + P_2 \sin \lambda, & q_2 &= Q_2 \cos \lambda + P_1 \sin \lambda, \\ p_1 &= -Q_2 \sin \lambda + P_1 \cos \lambda, & p_2 &= -Q_1 \sin \lambda + P_2 \cos \lambda. \end{aligned}$$

Given that the original Hamiltonian is

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} (q_1^2 + q_2^2 + p_1^2 + p_2^2),$$

determine the new Hamiltonian  $H(\mathbf{Q}, \mathbf{P})$ . Hence solve for the dynamics, subject to the constraints  $Q_2 = P_2 = 0$ .

7. A group of particles, all of the same mass  $m$ , have initial heights  $z_0$  and vertical momenta  $p_0$  lying in the rectangle  $-a \leq z_0 \leq a$ ,  $-b \leq p_0 \leq b$  in phase space. The particles fall freely in a uniform gravitational field for a time  $t$ . Find the region of phase space in which they lie at time  $t$ , and show by direct calculation that its area is still  $4ab$ .

8. A *Poisson structure* on  $\mathbb{R}^n$  is an antisymmetric matrix whose components  $\pi^{AB}(x)$  may depend on the coordinates  $x^A \in \mathbb{R}^n$ ,  $A = 1, \dots, n$ , such that the Poisson bracket

$$\{f, g\} = \sum_{AB} \pi^{AB}(x) \frac{\partial f}{\partial x^A} \frac{\partial g}{\partial x^B}$$

satisfies the Jacobi identity.

(a) Show that

$$\{fg, h\} = f\{g, h\} + \{f, h\}g.$$

(b) Assume that the matrix  $\pi$  is invertible, and suppose that its inverse is an anti-symmetric matrix  $\omega$  whose components  $\omega_{AB}(x)$  obey  $\pi^{AB}(x)\omega_{BC}(x) = \delta^A_C$  for all  $x$ . Show that  $\omega$  satisfies

$$\partial_A\omega_{BC} + \partial_C\omega_{AB} + \partial_B\omega_{CA} = 0$$

where  $\partial_A = \frac{\partial}{\partial x^A}$ . [Hint: Note that  $\pi^{AB} = \{x^A, x^B\}$ .]

(c) Set  $x^A = (x, y, z)$ . Show that

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y$$

defines a Poisson structure on  $\mathbb{R}^3$ , and find Hamilton's equations corresponding to a Hamiltonian  $H = \alpha x^2 + \beta y^2 + \gamma z^2$ , where  $\alpha$ ,  $\beta$  and  $\gamma$  are non-zero constants. Is this Poisson structure invertible?

9. Explain what is meant by an *adiabatic invariant* for a mechanical system with one degree of freedom.

A light string passes through a small hole in the roof of a lift compartment of a very high skyscraper, and a small weight is attached to the lower end. Initially, the lift is at rest and the system behaves like a simple pendulum executing small oscillations. Construct a Hamiltonian for the system and use the theory of adiabatic invariants to discuss what happens to the frequency, linear and angular amplitudes of the motion if:

- (a) the lift begins to move upwards with slowly increasing acceleration, with the string attached at the hole;
- (b) the lift stays at rest, but the string is slowly withdrawn through the roof.

10. Consider a system with Hamiltonian

$$H = \frac{p^2}{2m} + \lambda q^{2n},$$

where  $\lambda$  is a positive constant and  $n$  is a positive integer. Show that the action variable  $I$  and the energy  $E$  are related by

$$E = \lambda^{1/(n+1)} \left( \frac{n\pi I}{J_n} \right)^{2n/(n+1)} \left( \frac{1}{2m} \right)^{n/(n+1)},$$

where  $J_n = \int_0^1 (1-x)^{1/2} x^{(1-2n)/2n} dx$ .

Consider a particle that moves in a potential  $V(q) = \lambda q^4$ . Assuming that  $\lambda$  varies slowly with time, show that the particle's total energy  $E$  is proportional to  $\lambda^{1/3}$ . Conversely, in the case that  $\lambda$  is fixed, show that the period of the motion is proportional to  $(\lambda E)^{-1/4}$ .

11. A pulsar of mass  $m$  moves in a planar orbit around a luminous supergiant star with mass  $M \gg m$ . You may regard the supergiant as being fixed at the origin of a plane-polar coordinate system  $(r, \theta)$ , and the neutron star as moving in a central potential  $V(r) = -GMm/r$ . Construct the Hamiltonian for the motion, and show that  $p_\theta$  and the total energy  $E$  are constants of motion.

The neutron star is in a non-circular orbit with  $E < 0$ . Give an expression for the adiabatic invariant  $J(E, p_\theta, M)$  associated with the radial motion. The supergiant is steadily losing mass in a radiatively driven wind. Show that, over a long timescale, we have  $E \propto M^2$ .

Eventually the supergiant becomes a supernova, throwing off its outer layers on a short timescale, and leaving behind a remnant black hole of mass  $M/2$ . Explain why the theory of adiabatic invariants cannot be used to calculate the new orbit.

[You may find the following integral helpful:

$$\int_{r_1}^{r_2} \left[ \left(1 - \frac{r_1}{r}\right) \left(\frac{r_2}{r} - 1\right) \right]^{1/2} dr = \frac{\pi}{2}(r_1 + r_2) - \pi\sqrt{r_1 r_2},$$

where  $0 < r_1 < r_2$ .]

12. [Optional, based on 2010 Paper 4, Section II, Question 15D]

A system is described by the Hamiltonian  $H(q, p, t)$ . Define the *Poisson bracket*  $\{f, g\}$  of two functions  $f(q, p, t)$  and  $g(q, p, t)$ . Show from Hamilton's equations that

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

Consider the Hamiltonian

$$H = \frac{1}{2} (p^2 + \omega^2 q^2),$$

where  $\omega = \omega(t)$ , and define

$$a = \frac{p - i\omega q}{\sqrt{2\omega}}, \quad a^* = \frac{p + i\omega q}{\sqrt{2\omega}},$$

where  $i^2 = -1$ . Evaluate  $\{a, a\}$  and  $\{a, a^*\}$ , and show that  $\{a, H\} = -i\omega a$  and  $\{a^*, H\} = i\omega a^*$ . Show further that, when  $f(q, p, t)$  is regarded as a function of the independent complex variables  $(a, a^*)$  and of  $t$ , one has

$$\frac{df}{dt} = i\omega \left( a^* \frac{\partial f}{\partial a^*} - a \frac{\partial f}{\partial a} \right) - \frac{1}{2} \frac{\dot{\omega}}{\omega} \left( a \frac{\partial f}{\partial a^*} + a^* \frac{\partial f}{\partial a} \right) + \frac{\partial f}{\partial t}.$$

Deduce that, in the case  $d\omega/dt = 0$ , both  $(\log a^* - i\omega t)$  and  $(\log a + i\omega t)$  are constant during the motion.

Consider now the case in which  $\omega(t)$  varies slowly with time. Writing  $f = (H/\omega)$ , show that the time-average of  $(df/dt)$  over one period,  $(2\pi/\omega)$ , is approximately zero (that is, to order  $(\dot{\omega}^2, \ddot{\omega})$ ). [Hint: You might like to start by writing  $a = A(t)e^{-i\omega t} = A(0)e^{-i\omega t} + O(\dot{\omega})$ .]