Classical Dynamics
Lecturer: David Stuart

Mathematical Tripos, Part II
MT 2024

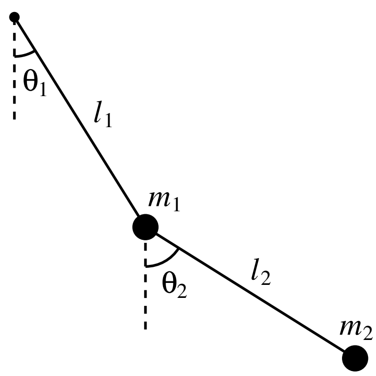
Example Sheet 1

1. A circular hoop of radius a lies in a vertical plane. The hoop rotates with constant angular velocity ω around a fixed vertical axis that goes through its centre, O. A bead of mass m is threaded on the hoop and moves without friction. Its location is denoted by A. The angle between the line OA and the downward vertical is $\psi(t)$.

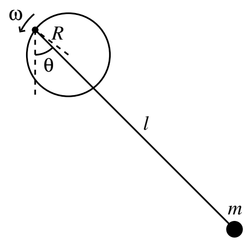
   (a) Using the Lagrangian formalism, derive a second-order differential equation for $\psi(t)$.

   (b) Assume now that the hoop rotates freely about the vertical axis without friction. Write down the Lagrangian of the system, neglecting the mass of the hoop. Find the additional conserved quantity.

2. A double pendulum is drawn below. Two light rods, of lengths l_1 and l_2, oscillate in the same plane. Attached to them are masses m_1 and m_2. How many degrees of freedom does the system have? Write down the Lagrangian describing the dynamics. Derive the equations of motion.



3. The pivot of a simple pendulum is attached to the rim of a disc of radius R, which rotates about its centre in the plane of the pendulum with constant angular velocity ω. (See the diagram below.) Write down the Lagrangian and derive the equation of motion for the dynamical variable θ.



4. A particle of mass m_1 is restricted to move on a circle of radius R_1 in the plane $z = 0$, with centre at $(x, y) = (0, 0)$. A second particle, of mass m_2, is restricted to move on a circle of radius R_2 in the plane $z = c$, with centre at $(x, y) = (0, a)$. The two particles are connected by a spring; the resulting potential energy is

$$V = \frac{1}{2}\omega^2 d^2\,,$$

where d is the distance between the particles.

(a) Identify the two generalized coordinates and write down the Lagrangian of the system.

(b) Write down the Lagrangian in the case that one circle lies directly beneath the other, $a = 0$, and identify a conserved quantity that appears in this case.

5. Two particles, each of mass m, are connected by a light rope of length l. One particle moves on a smooth horizontal table at a variable distance r from a hole, through which the rope is threaded. The second particle hangs beneath the table.

(a) Assume initially that the second particle hangs directly beneath the hole. Write down the Lagrangian of the system in terms of r and a variable ψ, describing the angle that the first particle makes with respect to a fixed axis. Identify an ignorable coordinate. Write down the equation of motion for the remaining coordinate, assuming that the rope remains taut.

(b) Assume now that the second particle oscillates beneath the table, as a spherical pendulum. How many degrees of freedom does the system now have? Write down the Lagrangian describing this motion, assuming that the rope remains taut at all times. How many ignorable coordinates are there?

6. An electron, of mass m and charge $-e$, moves in a magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})$. The Lagrangian for the motion is

$$L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - e\,\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r})\,.$$

Show that Lagrange's equations reproduce the Lorentz force law for the electron.

(a) With respect to cylindrical polar coordinates (r, θ, z), consider the vector potential

$$\mathbf{A} = \frac{f(r)}{r}\,\mathbf{e}_\theta\,,$$

where $\mathbf{e}_\theta$ is the unit vector in the θ direction. At some initial time, the electron is at a distance r_0 from the z axis; its velocity is then in the (r, z) plane. Show that the electron's angular velocity about the z axis is given by

$$\dot{\theta} = \frac{e}{mr^2}\left[f(r) - f(r_0)\right]\,.$$

(b) (Again, with respect to cylindrical polar coordinates.) Consider the (different) vector potential,

$$\mathbf{A} = rg(z)\,\mathbf{e}_\theta\,,$$

where $g(z) > 0$. Find two constants of the motion. The electron is projected from a point (r_0, θ_0, z_0) with velocity $(2er_0 g(z_0)/m)\,\mathbf{e}_\theta$. Show that the electron will then describe a circular orbit, provided that $g'(z_0) = 0$. Show that this orbit is stable against small translations in the z direction, provided that $g''(z_0) > 0$.

7. The Lagrangian for a relativistic point particle of mass m is

$$L = -mc^2\sqrt{1 - \frac{|\dot{\mathbf{r}}|^2}{c^2}} - V(\mathbf{r})\,,$$

where c is the speed of light. Derive the equation of motion, and show that it reduces to Newton's equation of motion in the limit $|\dot{\mathbf{r}}| \ll c$.

8. Consider a system with n dynamical degrees of freedom, and generalized coordinates denoted by q^a, with $a = 1, \ldots, n$. The most general form for a purely kinetic Lagrangian is

$$L = \frac{1}{2} g_{ab}(q^1, ..., q^n)\dot{q}^a \dot{q}^b\,, \tag{$*$}$$

where the summation convention is being used. The functions $g_{ab} = g_{ba}$ depend on the generalized coordinates. Assume that $\det(g_{ab}) \neq 0$ so that the inverse matrix g^{ab} exists (obeying $g^{ab}g_{bc} = \delta^a{}_c$). Show that Lagrange's equations for this system are given by

$$\ddot{q}^a + \Gamma^a_{bc}\dot{q}^b\dot{q}^c = 0\,, \tag{$\dagger$}$$

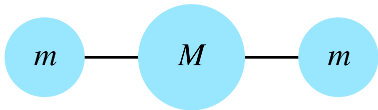
where you should define the objects Γ^a_{bc}, which depend on the coordinates q^d through g and its first derivatives. (You may assume they are symmetric in the lower indices, i.e., $\Gamma^a_{bc} = \Gamma^a_{cb}$.)

Write down a conserved quantity for the equations (†), and for the case $n = 4$ with $g_{11} = g_{22} = g_{33} = +1$ and $g_{44} = -1$ and compare with the concept of *proper time* from the *Dynamics and Relativity* notes. *Optional*: In the case $g_{11} = g_{22} = g_{33} = +1$ and $g_{44} = -(1 + 2V)$ with $V = V(q^1, q^2, q^3)$ and $\{\dot{q}^i\}_{i=1}^3$ very small, work out (†) to first order of approximation and interpret. (Working to first order means you can throw away terms quadratic in the small quantities $V, \partial_i V, \dot{q}^i$..... This calculation is behind the Newtonian approximation of general relativity, see pp. 77-79 in Weinberg's book *Gravitation and Cosmology*.)

Please send any comments and corrections to `dmas2@cam.ac.uk`

Example Sheet 2

1. The linear triatomic molecule drawn below consists of two identical outer atoms of mass m and a middle atom of mass M. It is a rough approximation to CO_2.



The interactions between neighbouring atoms are governed by a complicated potential $V(|\mathbf{r}_{i+1} - \mathbf{r}_i|)$. If we restrict attention to motion in the x direction parallel to the molecule, the Lagrangian is

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - V(x_2 - x_1) - V(x_3 - x_2)\,,$$

where x_i is the position of the i^{th} particle. Let $r_0 = |x_{i+1} - x_i|$ be the separation of neighbouring atoms in equilibrium. Write down the equation describing small deviations from equilibrium in terms of the masses and the quantity $k = V''(r_0)$. Show that the system has three normal modes and calculate the frequencies of oscillation of the system. One of these frequencies vanishes; what is the interpretation of this?

2. A horizontal square wire frame with vertices $ABCD$ and side length $2a$ rotates with constant angular velocity ω about a vertical axis through A. A bead of mass m is threaded on BC and moves without friction. The bead is connected to B and C by two identical light springs of spring constant k and equilibrium length a.

   (a) Introducing the displacement $\eta(t)$ of the particle from the midpoint of BC, determine the Lagrangian $L(\eta, \dot{\eta})$.

   (b) Derive the equation of motion and identify the constant of the motion.

   (c) Describe the motion of the bead. Find the condition for there to be a stable equilibrium and find the frequency of small oscillations about it when it exists.

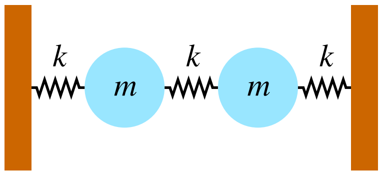
3. A pendulum consists of a mass m at the end of light rod of length l. The pivot of the pendulum is attached to a mass M, which is free to slide without friction along a horizontal rail. Take the generalized coordinates to be the position x of the pivot and the angle θ that the pendulum makes with the vertical.

   (a) Write down the Lagrangian and derive the equations of motion.

   (b) Find the non-zero frequency of small oscillations around the stable equilibrium.

   (c) Now suppose that a force acts on the the mass M, causing it to travel with constant acceleration a in the positive x direction. Find the equilibrium angle θ of the pendulum.

4. Two equal masses m are connected to each other and to fixed points by three identical springs of spring constant k as shown below. Write down the equations describing the motion of the system in the direction parallel to the springs. Find the normal modes and their frequencies.



5. Show that, for any solid body, the sum of any two principal moments of inertia is not less than the third. For what shapes is the sum of two equal to the third?

   Calculate the moments of inertia of:

   (a) A uniform solid sphere of mass M and radius R about a diameter.

   (b) A hollow sphere of mass M and radius R about a diameter.

   (c) A uniform solid circular cone of mass M, height h and base radius R with respect to the principal axes whose origin is at the vertex of the cone.

   (d) A uniform solid cylinder of radius R, height $2h$ and mass M about its centre of mass. For what height-to-radius ratio does the cylinder spin like a sphere?

   (e) A uniform solid ellipsoid of mass M and semi-axes a, b and c, defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1$$

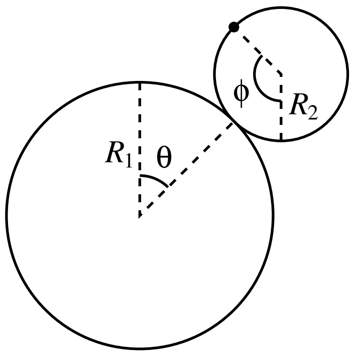
   with respect to the (x, y, z) axes with origin at the centre of mass. [*Hint:* With a change of coordinates, you can reduce this problem to that of the solid sphere.]

6. A cylindrical shell of radius R_2 rolls, without slipping, on a fixed cylinder of radius R_1 as shown below. Denote the angle through the centre of the first cylinder and the point of contact by θ. Denote the angle of a marked point on the upper cylinder with respect to a vertical axis by ϕ. Assume that the upper cylinder starts perched

near the top at $\theta = 0$, and that it rolls without slipping, acted upon by gravity. Show that the constraint for small θ is

$$R_1\theta = R_2(\phi - \theta)\,. \tag{$*$}$$

Is this constraint holonomic? Can the system be described by holonomic constraints for all θ? Write down the Lagrangian for the system assuming that this constraint holds. (Remember that the cylinder has kinetic energy both from the translation of its centre of mass and also from its spin.) Work out the equation of motion for θ. If the upper cylinder starts from rest at $\theta = 0$, show that it falls off the lower cylinder at $\theta = \pi/3$.



[*Note*: The question of when the cylinder falls off is not obviously captured by the Lagrangian you wrote down, which assumes the constraint $(*)$ holds. To solve this you will have to revert to Newtonian thinking and consider the constraint forces at play.]

7. A bead of mass 1 slides without friction on a fixed horizontal wire which occupies the interval $[-a, a]$ of the x-axis. A light spring, of spring constant 1, connects the bead to the point $-a$, and a second light spring, with the same spring constant, connects the bead to the point a. A massless rod of length 1 hangs freely from the bead and its other end carries a particle also of unit mass. The motion is restricted to the vertical plane containing the wire.

   (a) Show that the Lagrangian for the system is

   $$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(\dot{x}^2 + 2\cos\theta\,\dot{x}\dot{\theta} + \dot{\theta}^2) - x^2 + g\,\cos\theta\ ,$$

   where x is the position of the bead on the wire, θ is the angle between the rod and the downward vertical, and g is the acceleration due to gravity, and hence write down the equations of motion.

   (b) Put $g = 1$. Find any equilibrium points, and expand the Lagrangian to quadratic order about them, and hence determine their stability or otherwise (*optional*).

Please send any comments and corrections to `dmas2@cam.ac.uk`

Classical Dynamics
Lecturer: David Stuart

Mathematical Tripos, Part II
MT 2024

Example Sheet 3

1. *Inertia tensor*

(a) Prove that the principal moments of inertia, (I_1, I_2, I_3), are real and non-negative.

(b) During the lectures we outlined the proof of the Parallel Axis Theorem, which is a statement that the inertia tensor about a point P, which is displaced by $\mathbf{c}$ from the centre of mass C, is related to the inertia tensor about C by

$$I^P_{ab} = I^C_{ab} + M(c^2\delta_{ab} - c_a c_b)\,,$$

where M is the total mass of the body. Complete the proof of the theorem. (It will be helpful to choose the origin to be at the centre of mass.)

2. *Euler angles*

The rotation matrix that relates the body axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to the space axes $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is given in terms of the Euler angles $(\theta(t), \phi(t), \psi(t))$ by

$$\begin{aligned}\mathsf{R} &= \mathsf{R}_3(\psi)\mathsf{R}_{\mathrm{n}}(\theta)\mathsf{R}_z(\phi)\\ &= \begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}.\end{aligned}$$

(a) Calculate the matrix $\mathsf{A} = \dot{\mathsf{R}}\mathsf{R}^\top$ and verify that it is antisymmetric. By identifying A as the antisymmetric matrix associated with the angular velocity vector in the body frame, deduce that

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \begin{pmatrix}\dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi\\ -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi\\ \dot\psi + \dot\phi\cos\theta\end{pmatrix}.$$

(b) * [*optional extra*] Calculate the matrix $\mathsf{B} = \mathsf{R}^\top\dot{\mathsf{R}}$ and verify that it is antisymmetric. By identifying B as the antisymmetric matrix associated with the angular velocity vector in the space frame, deduce that

$$\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} = \begin{pmatrix}\dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi\\ \dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi\\ \dot\phi + \dot\psi\cos\theta\end{pmatrix}.$$

3. *Free symmetric top*

(a) Consider the torque-free motion of a round plate. Show that, in the body frame, the angular velocity vector $\boldsymbol{\omega}$ precesses around the body axis $\mathbf{e}_3$ perpendicular to the plate with constant angular frequency equal to ω_3.

(b) The physicist Richard Feynman tells the following story:

> "*I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.*
> *I had nothing to do, so I start figuring out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate – two to one. It came out of a complicated equation!*
> *I went on to work out equations for wobbles. Then I thought about how the electron orbits start to move in relativity. Then there's the Dirac equation in electrodynamics. And then quantum electrodynamics. And before I knew it....the whole business that I got the Nobel prize for came from that piddling around with the wobbling plate.*"

[Here the 'wobble' is associated with precession of the top about $\mathbf{e}_z$ (motion in ϕ), not nutation (motion in θ).]

Feynman was right about quantum electrodynamics. But what about the plate?

[You could try two alternative methods. First, by using the expression for ω_3 in terms of Euler's angles together with the result of part (a). Second, by writing down the Lagrangian of the top and deriving the equation of motion for θ.]

(c) Consider a uniform symmetric ellipsoid of mass M with semi-axes $a = b \neq c$ (see Example 2.5(e)). Find the ratio of the semi-axes for which $\dot{\phi}$, the angular frequency of precession of the top about the angular momentum $\mathbf{L}$, equals $\omega_3/(5\cos\theta)$. Deduce further that $\dot{\psi} = \frac{4}{5}\omega_3$. What is the relationship between $\dot{\phi}$ and $\dot{\psi}$ for small values of θ? Compare with the result obtained in part (b).

4. *Free asymmetric top (1)*

(a) Throw a book in the air. (Secure it with an elastic band first!) If the principal moments of inertia are $I_3 > I_2 > I_1$, convince yourself that the book can rotate in a stable manner about the principal axes $\mathbf{e}_1$ and $\mathbf{e}_3$, but not about $\mathbf{e}_2$.

(b) Use Euler's equations to show that the energy E and the squared angular momentum $|\mathbf{L}|^2$ of a free asymmetric top are conserved. Suppose that the initial conditions are such that

$$|\mathbf{L}|^2 = 2EI_2\,,$$

with the initial angular velocity $\boldsymbol{\omega}$ perpendicular to the intermediate principal axis $\mathbf{e}_2$. Show that $\boldsymbol{\omega}$ will ultimately end up parallel to $\mathbf{e}_2$. What is the characteristic timescale required to reach this steady state?

5. *Free asymmetric top (2)*

A rigid lamina (i.e. a two-dimensional object) has principal moments of inertia about the centre of mass given by

$$I_1 = (\mu^2 - 1)\,, \qquad I_2 = (\mu^2 + 1)\,, \qquad I_3 = 2\mu^2\,,$$

where $\mu > 1$. Write down Euler's equations for the lamina moving freely in space. Show that the component of the angular velocity in the plane of the lamina (i.e. $\sqrt{\omega_1^2 + \omega_2^2}$) is constant in time.

Choose the initial angular velocity to be $\boldsymbol{\omega} = \mu N \mathbf{e}_1 + N \mathbf{e}_3$. Define $\tan\alpha = \omega_2/\omega_1$, which is the angle the component of $\boldsymbol{\omega}$ in the plane of the lamina makes with $\mathbf{e}_1$. Show that it satisfies

$$\ddot{\alpha} + N^2 \cos\alpha \sin\alpha = 0$$

and deduce that, at time t,

$$\boldsymbol{\omega} = \mu N \,\mathrm{sech}(Nt)\,\mathbf{e}_1 + \mu N \tanh(Nt)\,\mathbf{e}_2 + N\,\mathrm{sech}(Nt)\,\mathbf{e}_3\,.$$

6. *Lagrange top*

Consider a heavy symmetric top of mass M, fixed at the point P which is a distance l from the centre of mass. The principal moments of inertia about P are (I_1, I_1, I_3) and the Euler angles are defined as in the lectures. The top is spun with initial conditions $\dot{\phi} = 0$ and $\theta = \theta_0$. Show that θ obeys the equation of motion

$$I_1 \ddot{\theta} = -\frac{dV_{\mathrm{eff}}}{d\theta}\,,$$

where

$$V_{\mathrm{eff}}(\theta) = \frac{I_3^2 \omega_3^2}{2I_1} \frac{(\cos\theta - \cos\theta_0)^2}{\sin^2\theta} + Mgl\cos\theta\,.$$

Suppose that the top is spinning very fast so that

$$I_3 \omega_3 \gg \sqrt{MglI_1}\,.$$

Show that the minimum of $V_{\mathrm{eff}}(\theta)$ is close to θ_0. Use this fact to deduce that the top nutates with angular frequency

$$\Omega \approx \frac{I_3}{I_1}\omega_3\,,$$

and sketch the subsequent motion.

7. *Lagrange top in Hamiltonian formalism*

The Lagrangian for the heavy symmetric top is

$$L = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 - Mgl \cos \theta \, .$$

Find the conjugate momenta p_θ, p_ϕ and p_ψ and the Hamiltonian $H(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi)$. Derive Hamilton's equations.

8. *Hamilton's equations*

A system with two degrees of freedom x and y has the Lagrangian

$$L = x\dot{y} + y\dot{x}^2 + \dot{x}\dot{y} \, .$$

Derive Lagrange's equations. Obtain the Hamiltonian $H(x, y, p_x, p_y)$. Derive Hamilton's equations and show that they are equivalent to Lagrange's equations.

Please send any comments and corrections to `dmas2@cam.ac.uk`

Example Sheet 4

1. Verify the Jacobi identity for Poisson brackets,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0\,.$$

2. A particle with mass m, position $\mathbf{r}$ and momentum $\mathbf{p}$ has angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Evaluate $\{x_i, L_j\}$, $\{p_i, L_j\}$, $\{L_i, L_j\}$ and $\{L_i, |\mathbf{L}|^2\}$.

   The Laplace–Runge–Lenz vector is defined as

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\,\hat{\mathbf{r}}\,,$$

   where k is a constant and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. Show that $\{L_i, A_j\} = \epsilon_{ijk} A_k$. For a system described by the Hamiltonian

$$H = \frac{|\mathbf{p}|^2}{2m} - \frac{k}{|\mathbf{r}|}\,,$$

   show, using Poisson brackets, that $\mathbf{A}$ is conserved.

3. A particle of charge q moves in a time-independent background magnetic field $\mathbf{B}$. Show that $\{m\dot{x}_i, m\dot{x}_j\} = q\epsilon_{ijk} B_k$ and $\{x_i, m\dot{x}_j\} = \delta_{ij}$.

   A *magnetic monopole* is a particle that produces a radial magnetic field of the form

$$\mathbf{B} = g\,\frac{\hat{\mathbf{r}}}{r^2}\,,$$

   where g is a constant and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. Consider a charged particle moving in the background of the magnetic monopole. Define the generalized angular momentum,

$$\mathbf{J} = m\,\mathbf{r} \times \dot{\mathbf{r}} - qg\,\hat{\mathbf{r}}\,.$$

   Show that $\{\mathbf{J}, H\} = \mathbf{0}$. Why does this imply that $\mathbf{J}$ is conserved?

4. In the lectures we constructed canonical transformations using generating functions. Consider canonical transformations $\mathbf{q} \mapsto \mathbf{Q}(\mathbf{q}, \mathbf{p})$, $\mathbf{p} \mapsto \mathbf{P}(\mathbf{q}, \mathbf{p})$ from the following perspective. Define the $2n$-dimensional vector $\mathbf{x} = (q_1, ..., q_n, p_1, ..., p_n)^\top$ and the $2n \times 2n$ matrix

$$\Omega = \begin{pmatrix} 0 & \mathtt{I}_n \\ -\mathtt{I}_n & 0 \end{pmatrix},$$

   where each entry is itself an $n \times n$ matrix.

   (a) Write Hamilton's equations for $\dot{\mathbf{x}}$ in terms of Ω and the Hamiltonian H.

(b) Hence deduce the following equation for the vector $\mathbf{X} = (Q_1, ..., Q_n, P_1, ..., P_n)^\top$:

$$\dot{\mathbf{X}} = \left(\mathsf{J}\Omega\mathsf{J}^\top\right) \frac{\partial H}{\partial \mathbf{X}} ,$$

where $J^A_B = \partial X^A / \partial x^B$ $(A, B = 1, \ldots, 2n)$ is the Jacobian matrix of the transformation. This implies that, if the Jacobian of a transformation satisfies

$$\mathsf{J}\Omega\mathsf{J}^\top = \Omega ,$$

then Hamilton's equations are invariant under that transformation. The transformations with such a Jacobian (said to be *symplectic*) are canonical.

(c) Use the above conclusion to prove that, if the Poisson bracket structure is preserved, then the transformation is canonical. Is it necessarily the case that a transformation which takes Hamiltonian equations into Hamiltonian equations is canonical?

5. Show that the following transformations are canonical:

(a) $P = \frac{1}{2}(p^2 + q^2) , \qquad Q = \arctan\left(\frac{q}{p}\right) ,$

(b) $P = \frac{1}{q} , \qquad Q = pq^2 ,$

(c) $P = 2\sqrt{q}\,(1 + \sqrt{q}\cos p)\sin p , \qquad Q = \log\left(1 + \sqrt{q}\cos p\right) .$

6. Show that the following transformation is canonical, for any constant λ:

$$\begin{aligned} q_1 &= Q_1 \cos\lambda + P_2 \sin\lambda , & q_2 &= Q_2 \cos\lambda + P_1 \sin\lambda , \\ p_1 &= -Q_2 \sin\lambda + P_1 \cos\lambda , & p_2 &= -Q_1 \sin\lambda + P_2 \cos\lambda . \end{aligned}$$

Given that the original Hamiltonian is

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(q_1^2 + q_2^2 + p_1^2 + p_2^2\right) ,$$

determine the new Hamiltonian $H(\mathbf{Q}, \mathbf{P})$. Hence solve for the dynamics, subject to the constraints $Q_2 = P_2 = 0$.

7. A group of particles, all of the same mass m, have initial heights z_0 and vertical momenta p_0 lying in the rectangle $-a \leqslant z_0 \leqslant a$, $-b \leqslant p_0 \leqslant b$ in phase space. The particles fall freely in a uniform gravitational field for a time t. Find the region of phase space in which they lie at time t, and show by direct calculation that its area is still $4ab$.

8. Explain what is meant by an *adiabatic invariant* for a mechanical system with one degree of freedom.

A light string passes through a small hole in the roof of a lift compartment of a very high skyscraper, and a small weight is attached to the lower end. Initially, the lift is at rest and the system behaves like a simple pendulum executing small oscillations. Construct a Hamiltonian for the system and use the theory of adiabatic invariants to discuss what happens to the frequency, linear and angular amplitudes of the motion if:

(a) the lift begins to move upwards with slowly increasing acceleration, with the string attached at the hole;

(b) the lift stays at rest, but the string is slowly withdrawn through the roof.

9. Consider a system with Hamiltonian

$$H = \frac{p^2}{2m} + \lambda\, q^{2n}\,,$$

where λ is a positive constant and n is a positive integer. Show that the action variable I and the energy E are related by

$$E = \lambda^{1/(n+1)} \left(\frac{n\pi I}{J_n}\right)^{2n/(n+1)} \left(\frac{1}{2m}\right)^{n/(n+1)}\,,$$

where $J_n = \int_0^1 (1-x)^{1/2} x^{(1-2n)/2n}\, dx$.

Consider a particle that moves in a potential $V(q) = \lambda\, q^4$. Assuming that λ varies slowly with time, show that the particle's total energy E is proportional to $\lambda^{1/3}$. Conversely, in the case that λ is fixed, show that the period of the motion is proportional to $(\lambda E)^{-1/4}$.

10. A pulsar of mass m moves in a planar orbit around a luminous supergiant star with mass $M \gg m$. You may regard the supergiant as being fixed at the origin of a plane-polar coordinate system (r, θ), and the neutron star as moving in a central potential $V(r) = -GMm/r$. Construct the Hamiltonian for the motion, and show that p_θ and the total energy E are constants of motion.

The neutron star is in a non-circular orbit with $E < 0$. Give an expression for the adiabatic invariant $J(E, p_\theta, M)$ associated with the radial motion. The supergiant is steadily losing mass in a radiatively driven wind. Show that, over a long timescale, we have $E \propto M^2$.

Eventually the supergiant becomes a supernova, throwing off its outer layers on a short timescale, and leaving behind a remnant black hole of mass $M/2$. Explain why the theory of adiabatic invariants cannot be used to calculate the new orbit.

[You may find the following integral helpful:

$$\int_{r_1}^{r_2} \left[\left(1 - \frac{r_1}{r}\right)\left(\frac{r_2}{r} - 1\right)\right]^{1/2} dr = \frac{\pi}{2}(r_1 + r_2) - \pi\sqrt{r_1 r_2}\,,$$

where $0 < r_1 < r_2$.]

11. [*Optional, based on 2010 Paper 4, Section II, Question 15D*]

A system is described by the Hamiltonian $H(q, p, t)$. Define the *Poisson bracket* $\{f, g\}$ of two functions $f(q, p, t)$ and $g(q, p, t)$. Show from Hamilton's equations that

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}\,.$$

Consider the Hamiltonian

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) \ ,$$

where $\omega = \omega(t)$, and define

$$a = \frac{p - i\omega q}{\sqrt{2\omega}}\,, \qquad a^* = \frac{p + i\omega q}{\sqrt{2\omega}}\,,$$

where $i^2 = -1$. Evaluate $\{a, a\}$ and $\{a, a^*\}$, and show that $\{a, H\} = -i\omega a$ and $\{a^*, H\} = i\omega a^*$. Show further that, when $f(q, p, t)$ is regarded as a function of the independent complex variables (a, a^*) and of t, one has

$$\frac{df}{dt} = i\omega\left(a^*\frac{\partial f}{\partial a^*} - a\frac{\partial f}{\partial a}\right) - \frac{1}{2}\frac{\dot\omega}{\omega}\left(a\frac{\partial f}{\partial a^*} + a^*\frac{\partial f}{\partial a}\right) + \frac{\partial f}{\partial t}\,.$$

Deduce that, in the case $d\omega/dt = 0$, both $(\log a^* - i\omega t)$ and $(\log a + i\omega t)$ are constant during the motion.

Consider now the case in which $\omega(t)$ varies slowly with time. Writing $f = (H/\omega)$, show that the time-average of (df/dt) over one period, $(2\pi/\omega)$, is approximately zero (that is, to order $(\dot\omega^2, \ddot\omega)$). [*Hint*: You might like to start by writing $a = A(t)e^{-i\omega t} = A(0)e^{-i\omega t} + O(\dot\omega)$.]

Please send any comments and corrections to `dmas2@cam.ac.uk`