Partial Differential Equations Example sheet 4

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4 Parabolic equations

In this section we consider parabolic operators of the form

$$Lu = \partial_t u + Pu$$

where

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u + cu \tag{4.1}$$

is an elliptic operator. Throughout this section $a_{jk} = a_{kj}, b_j, c$ are continuous functions, and

$$m\|\xi\|^2 \leq \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \leq M\|\xi\|^2 \tag{4.2}$$

for some positive constants m, M and all x, t and ξ. The basic example is the heat, or diffusion, equation $u_t - \Delta u = 0$, which we start by solving, first for x in an interval and then in $\mathbb{R}^n$. We then show that in both situations the solutions fit into an abstract framework of what is called a *semi-group of contraction operators*. We then discuss some properties of solutions of general parabolic equations (maximum principles and regularity theory).

4.1 The heat equation on an interval

Consider the one dimensional heat equation $u_t - u_{xx} = 0$ for $x \in [0,1]$, with Dirichlet boundary conditions $u(0,t) = 0 = u(1,t)$. Introduce the Sturm-Liouville operator $Pf = -f''$, with these boundary conditions. Its eigenfunctions $\phi_m = \sqrt{2}\sin m\pi x$ constitute an orthonormal basis for $L^2([0,1])$ (with inner product $(f,g)_{L^2} = \int f(x)g(x)dx$, considering here real valued functions). The eigenvalue equation is $P\phi_m = \lambda_m\phi_m$ with $\lambda_m = (m\pi)^2$. In terms of P the equation is:

$$u_t + Pu = 0$$

and the solution with initial data

$$u(0,x) = u_0(x) = \sum (\phi_m, u_0)_{L^2}\phi_m \,,$$

is given by

$$u(x,t) = \sum e^{-t\lambda_m} (\phi_m, u_0)_{L^2} \phi_m \,. \tag{4.3}$$

(In this expression $\sum$ means $\sum_{m=1}^{\infty}$.) An appropriate Hilbert space is to solve for $u(\cdot,t) \in L^2([0,1])$ given $u_0 \in L^2$, but the presence of the factor $e^{-t\lambda_m} = e^{-tm^2\pi^2}$ means the solution is far more regular for $t > 0$ than for $t = 0$:

Theorem 4.1.1 *Let* $u_0 = \sum (\phi_m, u_0)_{L^2} \phi_m$ *be the Fourier sine expansion of a function* $u_0 \in L^2([0,1])$. *Then the series* (4.3) *defines a smooth function* $u(x,t)$ *for* $t > 0$, *which satisfies* $u_t = u_{xx}$ *and* $\lim_{t\downarrow 0} \|u(x,t) - u_0(x)\|_{L^2} = 0$.

Proof Term by term differentiation of the series with respect to x, t has the effect only of multiplying by powers of m. For $t > 0$ the exponential factor $e^{-t\lambda_m} = e^{-tm^2\pi^2}$ thus ensures the convergence of these term by term differentiated series, absolutely and uniformly in regions $t \geq \theta > 0$ for any positive θ. It follows that for positive t the series defines a smooth function, which can be differentiated term by term, and which can be seen to solve $u_t = u_{xx}$. To prove the final assertion in the theorem, choose for each positive ϵ, a natural number $N = N(\epsilon)$ such that $\sum_{N+1}^{\infty} (\phi_m, u_0)_{L^2}^2 < \epsilon^2/4^2$. Let $t_0 > 0$ be such that for $|t| < t_0$

$$\| \sum_{1}^{N} (e^{-t\lambda_m} - 1)(\phi_m, u_0)_{L^2} \phi_m \|_{L^2} \leq \frac{\epsilon}{2} \,.$$

(This is possible because it is just a finite sum, each term of which has limit zero). Then the triangle inequality gives (for $0 < t < t_0$):

$$\begin{aligned} \|u(x,t) - u_0(x)\|_{L^2} &\leq \| \sum_{1}^{\infty} (e^{-t\lambda_m} - 1)(\phi_m, u_0)_{L^2} \phi_m \|_{L^2} \\ &\leq \frac{\epsilon}{2} + 2 \times \| \sum_{N+1}^{\infty} (\phi_m, u_0)_{L^2} \phi_m \|_{L^2} \leq \epsilon \end{aligned}$$

which implies that $\lim_{t\downarrow 0} \|u(x,t) - u_0(x)\|_{L^2} = 0$ since ϵ is arbitrary. (In the last bound, the restriction to t positive is crucial because it ensures that $e^{-t\lambda_m} \leq 1$.) □

The *instantaneous smoothing* effect established in this theorem is an important property of parabolic pde. In the next section it will be shown to occur for the heat equation on $\mathbb{R}^n$ also.

The formula (4.3) also holds, suitably modified, when P is replaced by any other Sturm-Liouville operator with orthonormal basis of eigenfunctions ϕ_m. For example, for if $Pu = -u''$ on $[-\pi, \pi]$) with periodic boundary conditions: in this case $\lambda_m = m^2$ and $\phi_m = e^{imx}/\sqrt{2\pi}$ for $m \in \mathbb{Z}$.

4.2 The heat kernel

The heat equation is $u_t = \Delta u$ where Δ is the Laplacian on the spatial domain. For the case of spatial domain $\mathbb{R}^n$ the distribution defined by the function

$$K(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi t^n}} \exp[-\frac{\|x\|^2}{4t}] & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases} \tag{4.4}$$

is the fundamental solution for the heat equation (in n space dimensions). This can be derived slightly indirectly: first using the Fourier transform (in the space variable x only) the following formula for the solution of the initial value problem

$$u_t = \Delta u\,, \qquad u(x,0) = u_0(x)\,, \qquad u_0 \in \mathcal{S}(\mathbb{R}^n)\,. \tag{4.5}$$

Let $K_t(x) = K(x,t)$ and let $*$ indicate convolution in the space variable only, then

$$u(x,t) = K_t * u_0(x) \tag{4.6}$$

defines for $t > 0$ a solution to the heat equation and by the approximation lemma (see question 2 sheet 3) $\lim_{t\to 0+} u(x,t) = u_0(x)$. Once this formula has been derived for $u_0 \in \mathcal{S}(\mathbb{R}^n)$ using the fourier transform it is straightforward to verify directly that it defines a solution for a much larger class of initial data, e.g. $u_0 \in L^p(\mathbb{R}^n)$, and the solution is in fact smooth for all positive t (*instantaneous smoothing*).

The *Duhamel principle* gives the formula for the inhomogeneous equation

$$u_t = \Delta u + F\,, \qquad u(x,0) = 0 \tag{4.7}$$

as $u(x,t) = \int_0^t U(x,t,s)ds$, where $U(x,t,s)$ is obtained by solving the family of homogeneous initial value problems:

$$U_t = \Delta U\,, \qquad U(x,s,s) = F(x,s)\,. \tag{4.8}$$

This gives the formula (with $F(x,t) = 0$ for $t < 0$)

$$u(x,t) = \int_0^t K_{t-s} * F(\cdot,s)\,ds = \int_0^t K_{t-s}(x-y)F(y,s)\,ds = K \circledast F(x,t)\,,$$

for the solution of (4.7), where $\circledast$ means space-time convolution.

4.3 Parabolic equations and semigroups

In this section we show that the solution formulae just obtained define semi-groups in the sense of definition 6.1.1.

Theorem 4.3.1 (Semigoup property - Dirichlet boundary conditions) *The solution operator for the heat equation given by* (4.3)

$$S(t) : u_0 \mapsto u(\cdot,t)$$

defines a strongly continuous one parameter semigroup *of contractions on the Hilbert space* $L^2([0,1])$.

Proof $S(t)$ is defined for $t \geq 0$ on $u \in L^2([0,1])$ by

$$S(t)\sum_m (\phi_m,u)_{L^2}\phi_m = \sum_m e^{-t\lambda_m}(\phi_m,u)_{L^2}\phi_m$$

and since $|e^{-t\lambda_m}| \leq 1$ for $t \geq 0$ and $\|u\|_{L^2}^2 = \sum_m(\phi_m,u)_{L^2}^2 < \infty$ this maps L^2 into L^2 and verifies the first two conditions in definition 6.1.1. The strong continuity condition (item 4 in definition 6.1.1) was proved in theorem 4.1.1. Finally, the fact that the $\{S(t)\}_{t\geq 0}$ are contractions on L^2 is an immediate consequence of the fact that $|e^{-t\lambda_m}| \leq 1$ for $t \geq 0$. □

To transfer this result to the heat kernel solution for whole space given by (4.6), note the following properties of the heat kernel:

- $K_t(x) > 0$ for all $t > 0, x \in \mathbb{R}^n$,
- $\int_{\mathbb{R}^n} K_t(x)dx = 1$ for all $t > 0$,
- $K_t(x)$ is smooth for $t > 0, x \in \mathbb{R}^n$, and for t fixed $K_t(\cdot) \in \mathcal{S}(\mathbb{R}^n)$,

the following result concerning the solution $u(\cdot, t) = S(t)u_0 = K_t * u_0$ follows from basic properties of integration (see appendix to §2 on integration):

- for $u_0 \in L^p(\mathbb{R}^n)$ the function $u(x,t)$ is smooth for $t > 0, x \in \mathbb{R}^n$ and satisfies $u_t - \Delta u = 0$,
- $\|u(\cdot,t)\|_{L^p} \leq \|u_0\|_{L^p}$ and $\lim_{t\to 0+} \|u(\cdot,t) - u_0\|_{L^p} = 0$ for $1 \leq p < \infty$.

From these and the approximation lemma (question 2 sheet 3) we can read off the theorem:

Theorem 4.3.2 (Semigroup property - $\mathbb{R}^n$) *(i) The formula $u(\cdot,t) = S(t)u_0 = K_t * u_0$ defines for $u_0 \in L^1$ a smooth solution of the heat equation for $t > 0$ which takes on the initial data in the sense that $\lim_{t\to 0+} \|u(\cdot,t) - u_0\|_{L^1} = 0$.*

(ii) The family $\{S(t)\}_{t\geq 0}$ also defines a strongly continuous semigroup of contractions on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

(iii) If in addition u_0 is continuous then $u(x,t) \to u_0(x)$ as $t \to 0+$ and the convergence is uniform if u_0 is uniformly continuous.

The properties of the heat kernel listed above also imply a maximum principle for the heat equation, which says that the solution always takes values in between the minimum and maximum values taken on by the intial data:

Lemma 4.3.3 (Maximum principle - heat equation on $\mathbb{R}^n$) *Let $u = u(x,t)$ be given by* (4.6). *If $a \leq u_0 \leq b$ then $a \leq u(x,t) \leq b$ for $t > 0, x \in \mathbb{R}^n$.*

Related maximum principle bounds hold for general second order parabolic equations, as will be shown in the next section.

4.4 The maximum principle

Maximum principles for parabolic equations are similar to the elliptic case, once the correct notion of boundary is understood. If $\Omega \subset \mathbb{R}^n$ is an open bounded subset with smooth boundary $\partial\Omega$ and for $T > 0$ we define $\Omega_T = \Omega \times (0, T]$ then the parabolic boundary of the space-time domain Ω_T is (by definition)

$$\partial_{par}\Omega_T = \overline{\Omega_T} - \Omega_T = \Omega \times \{t = 0\} \cup \partial\Omega \times [0, T]\,.$$

We consider variable coefficient parabolic operators of the form

$$Lu = \partial_t u + Pu$$

as in (4.1), still with the uniform ellipticity assumption (4.2) on P.

Theorem 4.4.1 *Let $u \in C(\overline{\Omega_T})$ have derivatives up to second order in x and first order in t which are continuous in Ω_T, and assume $Lu = 0$. Then*

- *if* $c = 0$ *(everywhere) then* $\max_{\overline{\Omega_T}} u(x,t) = \max_{\partial_{par}\Omega_T} u(x,t)$*, and*
- *if* $c \geq 0$ *(everywhere) then* $\max_{\overline{\Omega_T}} u(x,t) \leq \max_{\partial_{par}\Omega_T} u^+(x,t)$*, and* $\max_{\overline{\Omega_T}} |u(x,t)| = \max_{\partial_{par}\Omega_T} |u(x,t)|$.

where $u^+ = \max\{u, 0\}$ *is the positive part of the function* u*.*

Proof We prove the first case (when $c = 0$ everywhere). To prove the maximum principle bound, consider $u^\epsilon(x,t) = u(x,t) - \epsilon t$ which verifies, for $\epsilon > 0$, the strict inequality $Lu^\epsilon < 0$. First prove the result for u^ϵ:

$$\max_{\overline{\Omega_T}} u^\epsilon(x,t) = \max_{\partial_{par}\Omega_T} u^\epsilon(x,t)$$

Since $\partial_{par}\Omega_T \subset \overline{\Omega_T}$ the left side is automatically $\geq$ the right side. If the left side were strictly greater there would be a point (x_*, t_*) with $x_* \in \Omega$ and $0 < t_* \leq T$ at which the maximum value is attained:

$$u^\epsilon(x_*, t_*) = \max_{(x,t)\in\overline{\Omega_T}} u^\epsilon(x,t)\,.$$

By calculus first and second order conditions: $\partial_j u^\epsilon = 0$, $u_t^\epsilon \geq 0$ and $\partial_{ij}^2 u_x^\epsilon \leq 0$ (as a symmetric matrix - i.e. all eigenvalues are ≤ 0). These contradict $Lu^\epsilon < 0$ at the point (x_*, t_*). Therefore

$$\max_{\overline{\Omega_T}} u^\epsilon(x,t) = \max_{\partial_{par}\Omega_T} u^\epsilon(x,t)\,.$$

Now let $\epsilon \downarrow 0$ and the result follows. The proof of the second case is similar. □

4.5 Regularity for parabolic equations

Consider the Cauchy problem for the parabolic equation $Lu = \partial_t u + Pu = f$, where

$$Pu = -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n} b_j \partial_j u + cu \tag{4.9}$$

with initial data u_0. For simplicity assume that the coefficients are all smooth functions of $x, t \in \overline{\Omega_\infty}$. The weak formulation of $Lu = f$ is obtained by multiplying by a test function $v = v(x)$ and integrating by parts, leading to (where $(\,\cdot\,)$ means the L^2 inner product defined by integration over $x \in \Omega$):

$$(u_t\,,\, v\,) + B(u,v) = (f, v)\,, \tag{4.10}$$

$$B(u,v) = \int \big(\sum_{jk} a_{jk}\, \partial_j u \partial_k v + \sum b_j \partial_j u v \,+ cuv\big)\, dx\,.$$

To give a completely precise formulation it is necessary to define in which sense the time derivative u_t exists. To do this in a natural and general way requires the introduction of Sobolev spaces H^s for negative s - see §5.9 and §7.1.1-§7.1.2 in the book of Evans. However stronger assumptions on the initial data and inhomogeneous term are made a simpler statement is possible. (In the following statement $u(t)$ means the almost everywhere defined function of t taking values in a space of functions of x.)

Theorem 4.5.1 *For $u_0 \in H^1_0(\Omega)$ and $f \in L^2(\Omega_T)$ there exists*

$$u \in L^2([0,T]; H^2(\Omega) \cap L^\infty([0,T]; H^1_0(\Omega))$$

with time derivative $u_t \in L^2(\Omega_T)$ which satisfies (4.10) *for all $v \in H^1_0(\Omega)$ and almost every $t \in [0,T]$ and $\lim_{t\to 0+} \|u(t) - u_0\|_{L^2} = 0$. Furthermore it is unique and has the* parabolic regularity *property:*

$$\int_0^T (\|u(t)\|^2_{H^2(\Omega)} + \|u_t\|_{L^2(\Omega)})\, dt + \text{ess} \sup_{0\le t\le T} \|u(t)\|^2_{H^1_0(\Omega)} \le C(\|f\|_{L^2(\Omega_T)} + \|u_0\|_{H^1_0(\Omega)}) . \tag{4.11}$$

The time derivative is here to be understood in a weak/distributional sense as discussed in the sections of Evans' book just referenced, and the proof of the regularity (4.11) is in §7.1.3 of the same book. In the following result we will just verify that the bound holds for smooth solutions of the inhomogeneous heat equation on a periodic interval:

Theorem 4.5.2 *The Cauchy problem*

$$u_t - u_{xx} = f \,, \quad u(x,0) = u_0(x)$$

where $f = f(x,t)$ is a smooth function which is 2π-periodic in x, and the initial value u_0 is also smooth and 2π-periodic, admits a smooth solution for $t > 0$, 2π-periodic in x, which verifies the parabolic regularity estimate:

$$\int_0^T (\|u_t(t)\|^2_{L^2} + \|u(t)\|^2_{H^2})\, dt \le C\,(\|u_0\|^2_{H^1} + \int_0^T \int_{-\pi}^{\pi} |f(x,t)|^2\, dx dt) .$$

Here the norms inside the time integral are the Sobolev norms on 2π-periodic functions of x taken at fixed time.

Proof To prove existence, search for a solution in Fourier form, $u = \sum \hat{u}(m,t) e^{imx}$ and obtain the ODE

$$\partial_t \hat{u}(m,t) + m^2 \hat{u}(m,t) = \hat{f}(m,t) \,, \quad \hat{u}(m,0) = \hat{u}_0(m)$$

which has solution

$$\hat{u}(m,t) = e^{-m^2 t}\hat{u}_0(m) + \int_0^t e^{-m^2(t-s)} \hat{f}(m,s)\, ds \,.$$

Now by properties of Fourier series, $\hat{u}_0(m)$ is a rapidly decreasing sequence, and the same is true for $\hat{f}(m,t)$ locally uniformly in time, since

$$\max_{0\le t\le T} m^j |\hat{f}(m,t)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \max_{0\le t\le T} |\partial_x^j f(x,t)|\, dx \,.$$

Now, estimating $\hat{u}(m,t)$ for $0 \le t \le T$ simply as

$$|\hat{u}(m,t)| \le |\hat{u}_0(m)| + |T| \max_{0\le t\le T} |\hat{f}(m,t)| \,,$$

we see that $\hat{u}(m,t)$ is a rapidly decreasing sequence since $\hat{u}_0(m)$ and $\hat{f}(m,t)$ are. Differentiation in time just gives factors of m^2, and so $\partial_t^j \hat{u}(m,t)$ is also rapidly decreasing for each $j \in \mathbb{N}$. Therefore $u = \sum \hat{u}(m,t)e^{imx}$ defines a smooth function for positive time, and it verifies the equation (by differentiation through the sum, since this is allowed by rapidly decreasing property just established.)

To obtain the estimate, we switch to energy methods: multiply the equation by u_t and integrate. This leads to

$$\int_0^T \int_{-\pi}^{\pi} u_t^2 \, dxdt + \int_{-\pi}^{\pi} u_x^2 \, dx\big|_{t=T} = \int_{-\pi}^{\pi} u_x^2 \, dx\big|_{t=0} + \int_0^T \int_{-\pi}^{\pi} f u_t \, dxdt \, .$$

Using the Hölder inequality on the final term, this gives an estimate

$$\int_0^T \|u_t(t)\|_{L^2}^2 \, dt + \max_{0 \le t \le T} \|u(t)\|_{H^1}^2 \le C \left(\|u_0\|_{H^1}^2 + \int_0^T \int_{-\pi}^{\pi} |f(x,t)|^2 \, dxdt \right) .$$

(Here and below $C > 0$ is just a positive constant whose precise value is not important). To obtain the full parabolic regularity estimate from this, it is only necessary to use the equation itself to estimate

$$\int_0^T \|u_{xx}(t)\|_{L^2}^2 \, dt \le C \Big(\int_0^T \|u_t(t)\|_{L^2}^2 \, dt + \int_0^T \|f(x,t)\|_{L^2}^2 \, dt \Big) ,$$

and combining this with the previous bound completes the proof. □

The parabolic regularity estimate in this theorem can alternatively be derived from the Fourier form of the solution (exercise).

5 Hyperbolic equations

A second order equation of the form

$$u_{tt} + \sum_j \alpha_j \partial_t \partial_j u + Pu = 0$$

with P as in (4.1) (with coefficients potentially depending upon t and x), is strictly hyperbolic if the principal symbol

$$\sigma(\tau, \xi; t, x) = -\tau^2 - (\alpha \cdot \xi)\tau + \sum_{jk} a_{jk} \xi_j \xi_k$$

considered as a polynomial in τ has two distinct real roots $\tau = \tau_{\pm}(\xi; t, x)$ for all nonzero ξ. We will mostly study the wave equation

$$u_{tt} - \Delta u = 0 \, , \tag{5.12}$$

starting with some representations of the solution for the wave equation. In this section we write $u = u(t,x)$, rather than $u(x,t)$, for functions of space and time to fit in with the most common convention for the wave equation.

5.1 The one dimensional wave equation: general solution

Introducing characteristic coordinates $X_\pm = x \pm t$, the wave equation takes the form $\partial^2_{X_+X_-} u = 0$, which has general classical solution $F(X_-) + G(X_+)$, for arbitrary C^2 functions F, G (by calculus). Therefore, the general C^2 solution of $u_{tt} - u_{xx} = 0$ is

$$u(t,x) = F(x-t) + G(x+t)$$

for arbitrary C^2 functions F, G. (This can be proved by changing to the characteristic coordinates $X_\pm = x \pm t$, in terms of which the wave equation is $\frac{\partial^2 u}{\partial X_+ \partial X_-} = 0$.

From this can be derived the solution at time $t > 0$ of the inhomogeneous initial value problem:

$$u_{tt} - u_{xx} = f \tag{5.13}$$

with initial data

$$u(0,x) = u_0(x)\,, \qquad u_t(0,x) = u_1(x)\,. \tag{5.14}$$

$$u(t,x) = \frac{1}{2}\big(u_0(x-t) + u_0(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,dy + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\,dyds\,. \tag{5.15}$$

Notice that there is again a "Duhamel principle" for the effect of the inhomogeneous term since

$$\frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\,dyds \;=\; \int_0^t U(t,s,x)ds$$

where $U(t,s,x)$ is the solution of the *homogeneous* problem with data $U(s,s,x) = 0$ and $\partial_t U(s,s,x) = f(s,x)$ specified at $t = s$.

Theorem 5.1.1 *Assuming that* $(u_0, u_1) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$ *and that* $f \in C^1(\mathbb{R}\times\mathbb{R})$ *the formula* (5.14) *defines a* $C^2(\mathbb{R}\times\mathbb{R})$ *solution of the wave equation, and furthermore for each fixed time* t*, the mapping*

$$\begin{aligned} C^r \times C^{r-1} &\to C^r \times C^{r-1} \\ (u_0(\cdot), u_1(\cdot)) &\mapsto (u(t,\cdot), u_t(t,\cdot)) \end{aligned}$$

is continuous for each integer $r \geq 2$*. (Well-posedness in* $C^r \times C^{r-1}$*.)*

The final property stated in the theorem does not hold in more than one space dimension (question 7). This is the reason Sobolev spaces are more appropriate for the higher dimensional wave equation.

5.2 The one dimensional wave equation on an interval

Next consider the problem $x \in [0,1]$ with Dirichlet boundary conditions $u(t,0) = 0 = u(t,1)$. Introduce the Sturm-Liouville operator $Pf = -f''$, with these boundary conditions as in §4.1, its eigenfunctions being $\phi_m = \sqrt{2}\sin m\pi x$ with eigenvalues $\lambda_m = (m\pi)^2$. In terms of P the wave equation is:

$$u_{tt} + Pu = 0$$

and the solution with initial data

$$u(0,x) = u_0(x) = \sum \widehat{u_0}(m)\phi_m\,, \qquad u_t(0,x) = u_1(x) = \sum \widehat{u_1}(m)\phi_m\,,$$

is given by

$$u(t,x) = \sum_{m=1}^{\infty} \cos(t\sqrt{\lambda_m})\widehat{u_0}(m)\phi_m + \frac{\sin(t\sqrt{\lambda_m})}{\sqrt{\lambda_m}}\widehat{u_1}(m)\phi_m$$

with an analogous formula for u_t. Recall the definition of the Hilbert space $H_0^1((0,1))$ as the closure of the functions in $C_0^\infty((0,1))$[1] with respect to the norm given by $\|f\|_{H^1}^2 = \int_0^1 f^2 + f'^2\,dx$. In terms of the basis ϕ_m the definition is:

$$H_0^1((0,1)) = \{f = \sum \hat{f}_m\phi_m : \|f\|_{H^1}^2 = \sum_{m=1}^{\infty}(1+m^2\pi^2)|\hat{f}_m|^2 < \infty\}\,.$$

(In all these expressions $\sum$ means $\sum_{m=1}^{\infty}$.) As equivalent norm we can take $\sum \lambda_m|\hat{f}_m|^2$. An appropriate Hilbert space for the wave equation with these boundary conditions is to solve for $(u, u_t) \in X$ where $X = H_0^1 \oplus L^2$, and precisely we will take the following:

$$X = \{(f,g) = (\sum \hat{f}_m\phi_m, \sum \hat{g}_m\phi_m) : \|(f,g)\|_X^2 = \sum(\lambda_m|\hat{f}_m|^2 + |\hat{g}_m|^2) < \infty\}\,.$$

Now the effect of the evolution on the coefficients $\widehat{u}(m,t)$ and $\widehat{u_t}(m,t)$ is the map

$$\begin{pmatrix}\widehat{u}(m,t)\\ \widehat{u_t}(m,t)\end{pmatrix} \mapsto \begin{pmatrix}\cos(t\sqrt{\lambda_m}) & \frac{\sin(t\sqrt{\lambda_m})}{\sqrt{\lambda_m}}\\ -\sqrt{\lambda_m}\sin(t\sqrt{\lambda_m}) & \cos(t\sqrt{\lambda_m})\end{pmatrix}\begin{pmatrix}\widehat{u}(m,0)\\ \widehat{u_t}(m,0)\end{pmatrix} \tag{5.16}$$

Theorem 5.2.1 *The solution operator for the wave equation*

$$S(t) : \begin{pmatrix}u_0\\ u_1\end{pmatrix} \mapsto \begin{pmatrix}u(t,\cdot)\\ u_t(t,\cdot)\end{pmatrix}$$

defined by (5.16) *defines a strongly continuous* group *of unitary operators on the Hilbert space X, as in definition 6.3.1.*

5.3 The wave equation on $\mathbb{R}^n$

To solve the wave equation on $\mathbb{R}^n$ take the Fourier transform in the space variables to show that the solution is given by

$$u(t,x) = (2\pi)^{-n}\int \exp^{i\xi\cdot x}(\cos(t\|\xi\|)\widehat{u_0}(\xi) + \frac{\sin(t\|\xi\|)}{\|\xi\|}\widehat{u_1}(\xi))d\xi \tag{5.17}$$

[1] i.e. smooth functions which are zero outside of a closed set $[a,b] \subset (0,1)$

for initial values $u(0,x) = u_0(x), u_t(0,x) = u_1(x)$ in $\mathcal{S}(\mathbb{R}^n)$. The Kirchhoff formula arises from some further manipulations with the fourier transform in the case $n = 3$ and $u_0 = 0$ and gives the following formula

$$u(t,x) = \frac{1}{4\pi t} \int_{y:\|y-x\|=t} u_1(y)\, d\Sigma(y) \tag{5.18}$$

for the solution at time $t > 0$ of $u_{tt} - \Delta u = 0$ with initial data $(u, u_t) = (0, u_1)$. The solution for the inhomogeneous initial value problem with general Schwartz initial data u_0, u_1 can then be derived from the Duhamel principle, which takes the same form as in one space dimension (as explained in §5.1).

5.4 The energy identity and finite propagation speed

Lemma 5.4.1 (Energy identity) *If u is a C^2 solution of the wave equation* (5.12)*, then*

$$\partial_t\Big(\frac{u_t^2 + |\nabla u|^2}{2}\Big) + \partial_i\big(-u_t\partial_i u\big) = 0$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

From this and the divergence theorem some important properties follow:

Theorem 5.4.2 (Finite speed of propagation) *If $u \in C^2$ solves the wave equation* (5.12)*, and if $u(0,x)$ and $u_t(0,x)$ both vanish for $\|x - x_0\| < R$, then $u(t,x)$ vanishes for $\|x - x_0\| < R - |t|$ if $|t| < R$.*

Proof Notice that the energy identity can be written $\text{div}_{t,x}(e,p) = 0$, where

$$(e,p) = \Big(\frac{u_t^2 + |\nabla u|^2}{2}, -u_t\partial_1 u, \cdots - u_t\partial_n u\Big) \in \mathbb{R}^{1+n}\,.$$

Let $t_0 > 0$ and consider the backwards light cone with vertex (t_0, x_0), i.e. the set

$$\{(t,x) \in \mathbb{R}^{1+3} : t \leq t_0, \|x - x_0\| \leq t_0 - t\}\,.$$

The outwards normal to this at (t,x) is $\nu = \frac{1}{\sqrt{2}}(1, \frac{x-x_0}{\|x-x_0\|}) \in \mathbb{R}^{1+n}$, which satisfies $\nu \cdot (e,p) \geq 0$ by the Cauchy-Schwarz inequality. Integrating the energy identity over the region formed by intersecting the backwards light cone with the slab $\{(t,x) \in \mathbb{R}^{1+3} : 0 \leq t \leq t_1\}$, and using the divergence theorem then leads to $\int_{\|x-x_0\|\leq t_0-t_1} e(t_1,x)\,dx \leq \int_{\|x-x_0\|\leq t_0} e(0,x)\,dx$. This implies the result by choosing $R = t_0$. □

Theorem 5.4.3 (Regularity for the wave equation) *For initial data $u(0,x) = u_0(x)$ and $u_t(0,x) = u_1(x)$ in $\mathcal{S}(\mathbb{R}^n)$, the formula* (5.17) *defines a smooth solution of the wave equation* (5.12)*, which satisfies the energy conservation law*

$$\frac{1}{2}\int_{\mathbb{R}^n} u_t(t,x)^2 + \|\nabla u(t,x)\|^2\,dx = E = \textit{constant}\,.$$

Furthermore, at each fixed time t there holds:

$$\|(u(t,\cdot),u_t(t,\cdot))\|_{H^{s+1}\times H^s} \leq C\|(u_0(\cdot),u_1(\cdot))\|_{H^{s+1}\times H^s}\,, \qquad C>0 \tag{5.19}$$

for each $s\in\mathbb{Z}_+$. Thus the wave equation is well-posed in the Sobolev norms $H^{s+1}\times H^s$ and regularity is preserved when measured in the Sobolev L^2 sense.

Proof The fact that (5.17) defines a smooth function is a consequence of the theorems on the properties of the Fourier transform and on differentiation through the integral in §2, which is allowed by the assumption that the initial data are Schwartz functions. Given this, it is straightforward to check that (5.17) defines a solution to the wave equation. Energy conservation follows by integrating the identity in lemma 5.4.1. Energy conservation almost gives (5.19) for $s=0$. It is only necessary to bound $\|u(t,\cdot)\|^2_{L^2}$, which may be done in the following way. To start, using energy conservation, we have:

$$\Big|\frac{d}{dt}\|u\|^2_{L^2}\Big| = \big|2(u,u_t)_{L^2}\big| \leq \|u\|_{L^2}\|u_t\|_{L^2} \leq \sqrt{2E}\|u\|_{L^2}$$

This implies that $F_\epsilon(t)=(\epsilon+\|u(t,\cdot)\|^2_{L^2})^{\frac12}$ satisfies[2], for any positive ϵ

$$\dot F_\epsilon(t)\leq\sqrt{2E}$$

and hence $\|u(t,\cdot)\|_{L^2}\leq F_\epsilon(t)\leq(\epsilon+\|u(0,\cdot)\|^2_{L^2})^{\frac12}+\sqrt{2E}t$, for any $\epsilon>0$. This completes the derivation of (5.19) for $s=0$. The corresponding cases of (5.19) for $s=1,2\ldots$ are then derived by successively differentiating the equation, and applying the energy conservation law to the differentiated equation. □

Remark 5.4.4 *Well-posedness and preservation of regularity do not hold for the wave equation when measured in uniform norms $C^r\times C^{r-1}$, except in one space dimension, see question 7.*

Remark 5.4.5 *For initial data $(u_0,u_1)\in H^{s+1}\times H^s$ there is a distributional solution $(u(t,\cdot),u_t(t,\cdot))\in H^{s+1}\times H^s$ at each time, which can be obtained by approximation using density of C_0^∞ in the Sobolev spaces H^s and the well-posedness estimate* (5.19).

6 One-parameter semigroups and groups

If A is a bounded linear operator on a Banach space its norm is

$$\|A\| = \sup_{u\in X,u\neq 0}\frac{\|Au\|}{\|u\|}\,, \qquad \text{(operator or uniform norm)}.$$

This definition implies that if A,B are bounded linear operators on X then $\|AB\|\leq\|A\|\|B\|$.

[2]The ϵ is introduced to avoid the possibility of dividing by zero.

6.1 Definitions

Definition 6.1.1 *A one-parameter family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on a Banach space X forms a semigroup if*

1. *$S(0) = I$ (the identity operator) , and*
2. *$S(t+s) = S(t)S(s)$ for all $t, s \geq 0$ (semi-group property).*
3. *It is called a uniformly continuous semigroup if in addition to (1) and (2):*
$$\lim_{t\to 0+} \|S(t) - I\| = 0\,, \qquad \text{(uniform continuity).}$$
4. *It is called a strongly continuous (or C_0) semigroup if in addition to (1) and (2):*
$$\lim_{t\to 0+} \|S(t)u - u\| = 0\,, \forall u \in X \qquad \text{(strong pointwise continuity).}$$
5. *If $\|S(t)\| \leq 1$ for all $t \geq 0$ the semigroup $\{S(t)\}_{t\geq 0}$ is called a semigroup of contractions.*

Notice that in 3 the symbol $\|\cdot\|$ means the operator norm, while in 4 the same symbol means the norm on vectors in X. Also observe that uniform continuity is a stronger condition than strong continuity.

6.2 Semigroups and their generators

For ordinary differential equations $\dot{x} = Ax$, where A is an $n \times n$ matrix, the solution can be written $x(t) = e^{tA}x(0)$ and there is a $1-1$ corespondence between the matrix A and the semigroup $S(t) = e^{tA}$ on $\mathbb{R}^n$. In this subsection[3] we discuss how this generalizes.

Uniformly continuous semigroups have a simple structure which generalizes the finite dimensional case in an obvious way - they arise as solution operators for differential equations in the Banach space X:

$$\frac{du}{dt} + Au = 0\,, \qquad \text{for } u(0) \in X \text{ given}\,. \tag{6.20}$$

Theorem 6.2.1 *$\{S(t)\}_{t\geq 0}$ is a uniformly continuous semgroup on X if and only if there exists a unique bounded linear operator $A : X \to X$ such that $S(t) = e^{-tA} = \sum_{j=0}^{\infty}(-tA)^j/j!$. This semigroup gives the solution to* (6.20) *in the form $u(t) = S(t)u(0)$, which is continuously differentiable into X. The operator A is called the infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$.*

This applies to ordinary differential equations when A is a matrix. It is not very useful for partial differential equations because partial differential operators are unbounded, whereas in the foregoing theorem the infinitesimal generator was necessarily bounded. For example for the heat equation we need to take $A = -\Delta$, the laplacian defined on some appropriate Banach space of functions. Thus it is necessary to consider more general semigroups, in particular the strongly continuous semigroups. An unbounded linear operator A is a linear map from a linear subspace $D(A) \subset X$ into X (or more generally into another Banach space Y). The subspace $D(A)$ is called the domain of A. An unbounded linear operator $A : D(A) \to Y$ is said to be

[3] This subsection is for background information only

- *densely defined* if $\overline{D(A)} = X$, where the overline means closure in the norm of X, and

- *closed* if the graph $\Gamma_A = \{(u, Au)|_{u \in D(A)}\} \subset X \times Y$ is closed in $X \times Y$.

A class of unbounded linear operators suitable for understanding strongly continuous semigroups is the class of *maximal monotone* operators in a Hilbert space:

Definition 6.2.2 *1. A linear operator $A : D(A) \to X$ on a Hilbert space X is monotone if $(u, Au) \geq 0$ for all $u \in D(A)$.*

2. A monotone operator $A : D(A) \to X$ is maximal monotone if, in addition, the range of $I + A$ is all of X, i.e. if:

$$\forall f \in X\, \exists u \in D(A) : (I + A)u = f\,.$$

Maximal monotone operators are automatically densely defined and closed, and there is the following generalization of theorem 6.2.1:

Theorem 6.2.3 (Hille-Yosida) *If $A : D(A) \to X$ is maximal monotone then the equation*

$$\frac{du}{dt} + Au = 0\,, \qquad \textit{for } u(0) \in D(A) \subset X \textit{ given}\,, \tag{6.21}$$

admits a unique solution $u \in C([0, \infty); D(A)) \cap C^1([0, \infty); X)$ with the property that $\|u(t)\| \leq \|u(0)\|$ for all $t \geq 0$ and $u(0) \in D(A)$. Since $D(A) \subset X$ is dense the map $D(A) \ni u(0) \to u(t) \in X$ extends to a linear map $S_A(t) : X \to X$ and by uniqueness this determines a strongly continuous semigroup of contractions $\{S_A(t)\}_{t \geq 0}$ on the Hilbert space X. Often $S_A(t)$ is written as $S_A(t) = e^{-tA}$.

Conversely, given a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ of contractions on X, there exists a unique maximal monotone operator $A : D(A) \to X$ such that $S_A(t) = S(t)$ for all $t \geq 0$. The operator A is the infinitesimal generator of $\{S(t)\}_{t \geq 0}$ in the sense that $\frac{d}{dt} S(t)u = Au$ for $u \in D(A)$ and $t \geq 0$ (interpreting the derivative as a right derivative at $t = 0$).

6.3 Unitary groups and their generators

Semigroups arise in equations which are not necessarily time reversible. For equations which are, e.g. the Schrödinger and wave equations, each time evolution operator has an inverse and the semigroup is in fact a group. In this subsection[4] We give the definitions and state the main result.

Definition 6.3.1 *A one-parameter family of unitary operators $\{U(t)\}_{t \in \mathbb{R}}$ on a Hilbert space X forms a group of unitary operators if*

1. *$U(0) = I$ (the identity operator) , and*

2. *$U(t + s) = U(t)U(s)$ for all $t, s \in \mathbb{R}$ (group property).*

[4] In this subsection you only need to know definition 6.3.1. The remainder is for background information.

3. *It is called a strongly continuous (or C_0) group of unitary operators if in addition to (1) and (2):*

$$\lim_{t\to 0}\|U(t)u-u\|=0\,,\forall u\in X \qquad \textit{(strong pointwise continuity).}$$

A maximal monotone operator A which is symmetric (=hermitian), i.e. such that

$$(Au,v)=(u,Av) \qquad \text{for all } u,v \text{ in } D(A)\subset X \tag{6.22}$$

generates a one-parameter *group* of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$, often written $U(t)=e^{-itA}$, by solving the equation

$$\frac{du}{dt}+iAu=0\,, \qquad \text{for } u(0)\in D(A)\subset X \text{ given}\,. \tag{6.23}$$

It is useful to introduce the adjoint operator $A*$ via the Riesz representation theorem: first of all let

$$D(A^*)=\{u\in X:\ \text{the map } v\mapsto (u,Av) \text{ extends to a bounded linear functional } X\to\mathbb{C}\}$$

so that $D(A^*)$ is a linear space, and for $u\in D(A^*)$ there exists a vector w_u such that $(w_u,v)=(u,Av)$ (by Riesz representation). The map $u\to w_u$ is linear on $D(A^*)$ and so we can define an unbounded linear operator $A^*:D(A^*)\to X$ by $A^*u=w_u$, and since we started with a symmetric operator it is clear that $D(A)\subset D(A^*)$ and $A^*u=Au$ for $u\in D(A)$; the operator A^* is thus an extension of A.

Definition 6.3.2 *If $A:D(A)\to X$ is an unbounded linear operator which is symmetric and if $D(A^*)=D(A)$ then A is said to be self-adjoint and we write $A=A^*$.*

Theorem 6.3.3 *Maximal monotone symmetric operators are self-adjoint.*

Theorem 6.3.4 (Stone theorem) *If A is a self-adjoint operator the equation* (6.23) *has a unique solution for $u(0)\in D(A)$ which may be written $u(t)=U_A(t)u(0)$ with $\|u(t)\|=\|u(0\|$ for all $t\in\mathbb{R}$. It follows that the $U_A(t)$ extend uniquely to define unitary operators $X\to X$ and that $\{U_A(t)\}_{t\in\mathbb{R}}$ constitutes a strongly continuous group of unitary operators which are written $U_A(t)=e^{-itA}$.*

Conversely, given a strongly continuous group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ there exists a self-adjoint operator A such that $U(t)=U_A(t)=e^{-itA}$ for all $t\in\mathbb{R}$.

6.4 Worked problems

1. Let $C^\infty_{per}=\{u\in C^\infty(\mathbb{R}):u(x+2\pi)=u(x)\}$ be the space of smooth $2\pi-$ periodic functions of one variable.

   (i) For $f\in C^\infty_{per}$ show that there exists a unique $u=u_f\in C^\infty_{per}$ such that

$$-\frac{\partial^2 u}{\partial x^2}+u=f.$$

(ii) Show that $I_f[u_f + \phi] > I_f[u_f]$ for every $\phi \in C^\infty_{per}$ which is not identically zero, where $I_f : C^\infty_{per} \to \mathbb{R}$ is defined by

$$I_f[u] = \frac{1}{2}\int_{-\pi}^{+\pi} \big((\frac{\partial u}{\partial x})^2 + u^2 - 2f(x)u\big)dx.$$

(iii) Show that the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = f(x),$$

with initial data $u(0,x) = u_0(x) \in C^\infty_{per}$ has, for $t > 0$ a smooth solution $u(t,x)$ such that $u(t,\cdot) \in C^\infty_{per}$ for each fixed $t > 0$, and give a representation of this solution as a Fourier series in x. Calculate $\lim_{t\to+\infty} u(t,x)$ and comment on your answer in relation to (i).

(iv) Show that $I_f[u(t,\cdot)] \le I_f[u(s,\cdot)]$ for $t > s > 0$, and that $I_f[u(t,\cdot)] \to I_f[u_f]$ as $t \to +\infty$.

Answer (i) Any solution $u_f \in C^\infty_{per}$ can be represented as a Fourier series: $u_f = \sum \hat{u}_f(\alpha)e^{i\alpha x}$, as can f. Here $\alpha \in \mathbb{Z}$. The fourier coefficients are rapidly decreasing i.e. faster than any polynomial so it is permissible to differentiate through the sum, and substituting into the equation we find that the coefficients $\hat{u}_f(\alpha)$ are uniquely determined by f according to $(1+\alpha^2)\hat{u}_f(\alpha) = \hat{f}(\alpha)$, hence

$$u_f(x) = \sum \frac{\hat{f}(\alpha)}{1+\alpha^2}e^{i\alpha x}.$$

(Can also prove uniqueness by noting that if there were two solutions u_1, u_2 then the difference $u = u_1 - u_2$ would solve $-u_{xx} + u = 0$. Now multply by u and integrating by parts (using periodicity) - this implies that $\int u_x^2 + u^2 = 0$ which implies that $u = u_1 - u_2 = 0$.)

(ii) Calculate, using the equation satisfied by u_f and integration by parts, that

$$I_f[u_f + \phi] - I_f[u_f] = \frac{1}{2}\int_{\pi}^{\pi}(\phi_x^2 + \phi^2)dx > 0$$

for non-zero $\phi \in C^\infty_{per}$.

(iii) Expand the solution in terms of Fourier series and then substitute into the equation and use integrating factor to obtain that the solution is $u(t,x) = \sum \hat{u}(\alpha,t)e^{i\alpha x}$ where

$$\hat{u}(\alpha,t) = e^{-t(1+\alpha^2)}\hat{u}_0(\alpha) + \int_0^t e^{-(t-s)(1+\alpha^2)}\hat{f}(\alpha)ds.$$

Carry out the integral to deduce that

$$\hat{u}(\alpha,t) = \frac{\hat{f}(\alpha)}{1+\alpha^2} + e^{-t(1+\alpha^2)}\big(\hat{u}_0(\alpha) - \frac{\hat{f}(\alpha)}{1+\alpha^2}\big).$$

which implies that $\hat{u}(\alpha,t) \to \hat{u}_f(\alpha) = \frac{\hat{f}(\alpha)}{1+\alpha^2}$ as $t \to +\infty$, and further that $u(x,t) \to u_f(x)$ uniformly in x as $t \to +\infty$.

(iv) By (i) and (iii) we see that $u(x,t) = u_f(x) + \phi(x,t)$ where $\hat{\phi}(\alpha,t) = e^{-t(1+\alpha^2)}(\hat{u}_0(\alpha) - \hat{u}_f(\alpha))$. Now apply (ii) and use the Parseval theorem to deduce that

$$\begin{aligned} I_f[u(t,\cdot)] - I_f[u_f] &= \pi\sum(1+\alpha^2)|\hat{\phi}(\alpha,t)|^2 \\ &= \pi\sum(1+\alpha^2)e^{-2t(1+\alpha^2)}|\hat{u}_0(\alpha) - \hat{u}_f(\alpha)|^2 \end{aligned}$$

which decreases to zero since $\hat{u}_0(\alpha)$ and $\hat{u}_f(\alpha)$ are rapidly decreasing.

2. For the equation $u_t - u_{xx} + u = f$, where $f = f(x,t)$ is a smooth function which is 2π-periodic in x, and the initial data $u(x,0) = u_0(x)$ are also smooth and 2π-periodic obtain the solution as a Fourier series $u = \sum \hat{u}(m,t)e^{imx}$ and hence verify the parabolic regularity estimate:

$$\int_0^T (\, \|u_t(t)\|_{L^2}^2 + \|u(t)\|_{H^2}^2 \,)\, dt \le C\, (\, \|u_0\|_{H^1}^2 + \int_0^T \int_{-\pi}^{\pi} |f(x,t)|^2 \, dxdt \,)\,.$$

Answer: Use the Fourier form of the solution $u(x,t) = \sum_{m\in\mathbb{Z}^n} \hat{u}(m,t)e^{im\cdot x}$ at each time t, and similarly for f, and the definition

$$H^s_{per} = \{u = \sum_{m\in\mathbb{Z}^n} \hat{u}(m)e^{im\cdot x} \in L^2 : \|u\|_{H^s}^2 = \sum_{m\in\mathbb{Z}^n} (1+\|m\|^2)^s |\hat{u}(m)|^2 < \infty\}\,,$$

is for the Sobolev spaces of fixed time functions 2π-periodic in each co-ordinate x_j and for $s = 0, 1, 2, \ldots$. Writing $\omega_m = 1 + \|m\|^2$, and using an integrating factor the solution is given by:

$$\hat{u}(m,t) = e^{-\omega_m t}\hat{u}(m,0) + \int_0^t e^{-(t-s)\omega_m}\, \hat{f}(m,s)\, ds$$

in Fourier representation. The second term is a convolution, so by the Hausdorff-Young inequality $\|f * g\|_{L^2}^2 \le \|f\|_{L^1}^2 \|g\|_{L^2}^2$ we obtain:

$$\begin{aligned}\int_0^T |\int_0^t e^{-(t-s)\omega_m}\, \hat{f}(m,s)\, ds\,|^2\, dt \;&\le\; \Big(\int_0^T |e^{-t\omega_m}|dt\Big)^2 \int_0^T |\hat{f}(m,t)|^2 dt \\ &\le\; \frac{1}{\omega_m^2} \int_0^T |\hat{f}(m,t)|^2 dt\,.\end{aligned}$$

Here we have made use of $\int_0^T e^{-\omega_m t}\, dt = \frac{1-e^{-\omega_m T}}{\omega_m} \le \frac{1}{\omega_m}$. Using this bound, and $|a+b|^2 \le 2(a^2+b^2)$, we obtain:

$$\begin{aligned}\int_0^T \omega_m^2 |\hat{u}(m,t)|^2\, dt &\le 2\Big[\int_0^T e^{-2t\omega_m}\, dt\, \omega_m^2\, |\hat{u}(m,0)|^2 \;+\; \int_0^T |\hat{f}(m,t)|^2 dt\Big] \\ &\le 2\Big[\, \frac{\omega_m}{2}\, |\hat{u}(m,0)|^2 \;+\; \int_0^T |\hat{f}(m,t)|^2 dt\,\Big]\,.\end{aligned}$$

Now sum over $m \in \mathbb{Z}^n$ and use the Parseval theorem and definitions of $\|\cdot\|_{H^s}$ to obtain

$$\int_0^T \|u(t)\|_{H^2}^2\, dt \le \; const. \Big[\|u(0)\|_{H^1}^2 \;+\; \int_0^T |f(t)|_{L^2}^2\, dt\Big]\,.$$

To obtain the inequality as stated it is sufficient to use the equation to obtain the same bound for $\int_0^T \|u_t(t)\|_{L^2}^2\, dt$ (with another constant).

3. (i) Define the Fourier transform $\hat{f} = \mathcal{F}(f)$ of a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, and also of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$.

(ii) From your definition compute the Fourier transform of the distribution $W_t \in \mathcal{S}'(\mathbb{R}^3)$ given by

$$W_t(\psi) = < W_t, \psi > = \frac{1}{4\pi t} \int_{\|y\|=t} \psi(y) d\Sigma(y)$$

for every Schwartz $\psi \in \mathcal{S}(\mathbb{R}^3)$. (Here $d\Sigma(y) = t^2 d\Omega(y)$ is the integration element on the sphere of radius t,) and hence deduce a formula (Kirchoff) for the solution of the initial value problem for the wave equation in three space dimensions,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0,$$

with initial data $u(0,x) = 0$ and $\frac{\partial u}{\partial t}(0,x) = g(x), x \in \mathbb{R}^3$ where $g \in \mathcal{S}(\mathbb{R}^3)$. Explain briefly why the formula is valid for arbitrary smooth f.

(iii) Show that any C^2 solution of the initial value problem in (ii) is given by the formula derived in (ii) (uniqueness).

(iv) Show that any two solutions of the initial value problem for

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u = 0,$$

with identical initial data as in (ii), also agree for any $t > 0$.

Answer (i) $\hat{f}(\xi) = \int f(x) e^{-ix\cdot\xi}\, dx$, and $\langle \hat{u}, f \rangle = \langle u, \hat{f} \rangle$. This defines $u \in \mathcal{S}'(\mathbb{R}^n)$ since for any $f \in \mathcal{S}(\mathbb{R}^n)$ the Fourier transform $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ also; in fact $f \mapsto \hat{f}$ is a linear homeomorphism on $\mathcal{S}(\mathbb{R}^n)$.

(ii) Compute

$$< W_t, \hat{f} >= \frac{1}{4\pi t} \int_{\|y\|=t} \hat{f}(y) d\Sigma(y) = \frac{t}{4\pi} \int_{\mathbb{R}^n} f(\xi) \int_{\|\Omega\|=1} e^{-it\|\xi\| \cos\theta}\, d\Omega d\xi$$

Here we are writing $\Omega = (\theta, \phi)$ for the spherical polar angles for y, with the direction of ξ taken as the "e_3 axis", so that $y \cdot \xi = \|\xi\| \|y\| \cos\theta = t\|\xi\| \cos\theta$. The inner integral can be performed, after inserting $d\Omega = \sin\theta d\theta d\phi$, and equals $2\pi \times (2 \sin t\|\xi\|)/(t\|\xi\|)$, so that overall:

$$< \hat{W}_t\, ,\, f >=< W_t\, ,\, \hat{f} >= \int_{\mathbb{R}^n} \frac{\sin t\|\xi\|}{\|\xi\|}\, f(\xi)\, d\xi\, .$$

This means $\hat{W}_t$ is the distribution determined by the function $\frac{\sin(t\|\xi\|)}{\|\xi\|}$. (This function is actually smooth and bounded by the Taylor expansion, and so determines a tempered distribution.)

But in Fourier variables the solution of the wave equation is:

$$\hat{u}(t,\xi) = \Big(\cos(t\|\xi\|) \widehat{u_0}(\xi) + \frac{\sin(t\|\xi\|)}{\|\xi\|} \widehat{u_1}(\xi) \Big)$$

for initial values $u(0,x) = u_0(x), u_t(0,x) = u_1(x)$ in $\mathcal{S}(\mathbb{R}^n)$. Comparing with the formula just derived, and applying the convolution theorem, it follows that the solution with $u_0 = 0$ and $u_1 = g$ is given at each time t by $u(t,\cdot) = W_t * g$, since then

$$\hat{u}(t,\xi) = \hat{W}_t(\xi) \hat{g}(\xi) = \frac{\sin(t\|\xi\|)}{\|\xi\|} \hat{g}(\xi)$$

(iii) Classical solutions of the wave equation obey the energy momentum conservation law

$$e_t + \nabla \cdot p = 0$$

where $e = (u_t^2 + |\nabla u|^2)/2$ and $p = -u_t \nabla u$. Integrate $e_t + \nabla \cdot p = 0$ over the part of the backward light cone with vertex (t_0, x_0), for some $t_0 > 0$, which lies in the slab between $\{t = 0\}$ and $\{t = t_1 < t_0\}$; i.e. the region

$$K_{t_0,x_0} = \{(t,x) \in \mathbb{R}^{1+3} : 0 \le t \le t_1, \|x - x_0\| \le t_0 - t\}\,.$$

Applying the divergence theorem, and noticing that if ν is the outward pointing normal on the sloping part of the boundary of this region, then $\nu \cdot (e,p) \ge 0$ by the Cauchy-Schwarz inequality, we deduce that

$$\int_{\|x-x_0\| \le t_0 - t_1} e(t_1, x)\, dx \le \int_{\|x-x_0\| \le t_0} e(0,x)\, dx\,. \tag{6.24}$$

This implies that if the initial data are zero then the solution is zero at all later times. By time reversal symmetry an identical argument implies the same thing for negative times. Applied to the difference of two solutions this implies uniqueness (since by linearity the difference of two solutions of the wave equation also solves the wave equation), and hence that any classical C^2 solution is given by the same formula as was derived in (ii).

(iv) Do essentially the same calculation as in (iii) but using this time that

$$e_t + \nabla \cdot p \,=\, -u_t^2 \le 0$$

which gives the same conclusion 6.24 for positive times. (However, since time reversal symmetry no longer holds, the argument cannot now be simply reversed to obtain the analogous inequality for negative times).

4. Consider a *continuous* function $t \mapsto u(t) \in \mathbb{C}$ such that $|u(t)| = 1\, \forall t$ and $u(t+s) = u(t)u(s)$ for all real s, t. Prove that there exists $a \in \mathbb{R}$ such that $u(t) = e^{iat}$. Deduce Stone's theorem on the Hilbert space $\mathbb{C}$.

   Answer There exists $\lambda(t)$, defined mod 2π, such that $u(t) = e^{i\lambda(t)}$. By continuity there exists $\delta > 0$ such that for $|t| \in I = (-\delta, +\delta)$ we have $|u(t) - 1| < \frac{1}{2}$. In this interval I, there is a unique $\lambda(t) \in (-\pi, +\pi)$ which is continuous and satisfies $u(t) = e^{i\lambda(t)}$ and $\lambda(t+s) = \lambda(t) + \lambda(s)$ for $s, t, s+t$ all in I. Let N be any integer sufficiently large that $\frac{1}{N} \in I$, and define $a = N\lambda(\frac{1}{N})$. Then the semigroup property implies that $u(\frac{m}{N}) = (u(\frac{1}{N}))^m = e^{im\lambda(\frac{1}{N})} = e^{ia\frac{m}{N}}$, for any integer m. The value of a thus defined is independent of N chosen as above: indeed, if a', N' were another such value we would also have $u(t) = e^{ia'\frac{m}{N'}}$ for all integral m. Clearly $\frac{1}{NN'} \in I$, so also defining $b = NN'\lambda(\frac{1}{NN'})$ we have by the additivity of λ that $a = b = a'$. Therefore a is unique, for all such integers N with $\frac{1}{N} \in I$ and so $u(t) = e^{ia\frac{m}{N}}$ for all such N and all $m \in \mathbb{Z}$. It follows from the density of $\frac{m}{N}$ in $\mathbb{R}$ and the continuity of $u(t)$ that $u(t) = e^{iat}$ for all $t \in \mathbb{R}$.

6.5 Example sheet 4

1. (a) Use the change of variables $v(t,x) = e^t u(t,x)$ to obtain an "x-space" formula for the solution to the initial value problem:

$$u_t + u = \Delta u \qquad u(0,\,\cdot\,) = u_0(\cdot) \in \mathcal{S}(\mathbb{R}^n).$$

Hence show that $|u(t,x)| \le \sup_x |u_0(x)|$ and use this to deduce well-posedness in the supremum norm (for $t > 0$ and all x).

If $a \le u_0(x) \le b$ for all x what can you say about the possible values of $u(t,x)$ for $t > 0$.

(b) Use the Fourier transform in x to obtain a (Fourier space) formula for the solution of:

$$u_{tt} - 2u_t + u = \Delta u \qquad u(0,\,\cdot\,) = u_0(\cdot) \in \mathcal{S}(\mathbb{R}^n),\ u_t(0,\cdot) = u_1(\cdot) \in \mathcal{S}(\mathbb{R}^n).$$

2. Show that if $u \in C([0,\infty)\times\mathbb{R}^n) \cap C^2((0,\infty)\times\mathbb{R}^n)$ satisfies (i) the heat equation, (ii) $u(0,x) = 0$ and (iii) $|u(t,x)| \le M$ and $|\nabla u(t,x)| \le N$ for some M, N then $u \equiv 0$. (Hint: multiply heat equation by $K_{t_0-t}(x-x_0)$ and integrate over $|x| < R, a < t < b$. Apply the divergence theorem, carefully let $R \to \infty$ and then $b \to t_0$ and $a \to 0$ to deduce $u(t_0,x_0) = 0$.)

3. Show that if $S(t)$ is a strongly continuous semigroup of contractions on a Banach space X with norm $\|\cdot\|$, then

$$\lim_{t\to 0+} \|S(t_0+t)u - S(t_0)u\| = 0\,, \qquad \forall u \in X \text{ and } \forall t_0 > 0\,.$$

4. Let $Pu = -(pu')' + qu$, with p and q smooth, be a Sturm-Liouville operator on the unit interval $[0,1]$ and assume there exist constants m, c_0 such that $p \ge m > 0$ and $q \ge c_0 > 0$ everywhere, and consider Dirichlet boundary conditions $u(0) = 0 = u(1)$. Assume $\{\phi_n\}_{n=1}^{\infty}$ are smooth functions which constitute an orthonormal basis for $L^2([0,1])$ of eigenfunctions: $P\phi_n = \lambda_n\phi_n$. Show that there exists a number $\gamma > 0$ such that $\lambda_n \ge \gamma$ for all n. Write down the solution to the equation $\partial_t u + Pu = 0$ with initial data $u_0 \in L^2([0,1])$ and show that it defines a strongly continuous semigroup of contractions on $L^2([0,1])$, and describe the large time behaviour.

5. (i) Let $\partial_t u_j + Pu_j = 0\,,\ j = 1,2$ where P is as in (4.1) and the functions u_j have the regularity assumed in theorem 4.4.1 and satisfy Dirichlet boundary conditions: $u_j(x,t) = 0\,\forall x \in \partial\Omega, t \ge 0$. Assuming, in addition to (4.2), that

$$c \ge c_0 > 0 \tag{6.25}$$

for some positive constant c_0 prove that for all $0 \le t \le T$:

$$\sup_{x\in\Omega} |u_1(x,t) - u_2(x,t)| \le e^{-tc_0} \sup_{x\in\Omega} |u_1(x,0) - u_2(x,0)|.$$

(ii) In the situation of part (i) with

$$Pu = -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n} b_j\partial_j u + cu\,, \tag{6.26}$$

assuming in addition to (4.2) and (6.25) also that a_{jk}, b_j are C^1 and that

$$\sum_{j=1}^{n} \partial_j b_j = 0\,, \quad \text{in } \overline{\Omega_T}\,,$$

prove that for all $0 \le t \le T$:

$$\int_\Omega |u_1(x,t) - u_2(x,t)|^2\,dx \le e^{-2tc_0} \int_\Omega |u_1(x,0) - u_2(x,0)|^2\,dx.$$

6. (i) Let K_t be the heat kernel on $\mathbb{R}^n$ at time t and prove directly by integration that

$$K_t * K_s = K_{t+s}$$

for $t, s > 0$ (semi-group property). Use the Fourier transform and convolution theorem to give a second simpler proof.
(ii) Deduce that the solution operators $S(t) = K_t *$ define a strongly continuous semigroup of contractions on $L^p(\mathbb{R}^n)\, \forall p < \infty$.
(iii) Show that the solution operator $S(t) : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ for the heat initial value problem satisfies $\|S(t)\|_{L^1 \to L^\infty} \leq ct^{-\frac{n}{2}}$ for positive t, or more explicitly, that the solution $u(t) = S_t u(0)$ satisfies $\|u(t)\|_{L^\infty} \leq ct^{-\frac{n}{2}} \|u(0)\|_{L^1}$, or:

$$\sup_x |u(x,t)| \leq ct^{-n/2} \int |u(x,0)| dx$$

for some positive number c, which should be found.
(iii) Now let $n = 4$. Deduce, by considering $v = u_t$, that if the inhomogeneous term $F \in \mathcal{S}(\mathbb{R}^4)$ is a function of x only, the solution of $u_t - \Delta u = F$ with zero initial data converges to some limit as $t \to \infty$. Try to identify the limit.

7. (i) Let $u(t,x)$ be a twice continuously differentiable solution of the wave equation on $\mathbb{R} \times \mathbb{R}^n$ for $n = 3$ which is radial, i.e. a function of $r = \|x\|$ and t. By letting $w = ru$ deduce that u is of the form

$$u(t,x) = \frac{f(r-t)}{r} + \frac{g(r+t)}{r}.$$

(ii) Show that the solution with initial data $u(0,\cdot) = 0$ and $u_t(0,\cdot) = G$, where G is radial and even function, is given by

$$u(t,r) = \frac{1}{2r} \int_{r-t}^{r+t} \rho G(\rho) d\rho.$$

(iii) Hence show that for initial data $u(0,\cdot) \in C^3(\mathbb{R}^n)$ and $u_t(0,\cdot) \in C^2(\mathbb{R}^n)$ the solution $u = u(t,x)$ need only be in $C^2(\mathbb{R} \times \mathbb{R}^n)$. Contrast this with the case of one space dimension.

8. Write down the solution of the Schrodinger equation $u_t = iu_{xx}$ with 2π-periodic boundary conditions and initial data $u(x,0) = u_0(x)$ smooth and 2π-periodic in x, and show that the solution determines a strongly continuous group of unitary operators on $L^2([-\pi,\pi])$. Do the same for Dirichlet boundary conditions i.e. $u(-\pi,t) = 0 = u(\pi,t)$ for all $t \in \mathbb{R}$.

9. (i) Write the one dimensional wave equation $u_{tt} - u_{xx} = 0$ as a first order in time evolution equation for $U = (u, u_t)$.

   (ii) Use Fourier series to write down the solution with initial data $u(0,\cdot) = u_0$ and $u_t(0,\cdot) = u_1$ which are smooth 2π-periodic and have zero mean: $\hat{u}_j(0) = 0$.

   (iii) Show that $\|u\|^2_{\dot{H}^1_{per}} = \sum_{m \neq 0} |m|^2 |\hat{u}(m)|^2$ defines a norm on the space of smooth 2π-periodic functions with zero mean. The corresponding complete Sobolev space is the case $s = 1$ of

$$\dot{H}^s_{per} = \{ \sum_{m \neq 0} \hat{u}(m) e^{im\cdot x} : \|u\|^2_{\dot{H}^s_{per}} = \sum_{m \neq 0} |m|^{2s} |\hat{u}(m)|^2 < \infty \},$$

the Hilbert space of zero mean 2π-periodic H^s functions.

(iv) Show that the solution defines a group of unitary operators in the Hilbert space

$$X = \{U = (u, v) \; : \; u \in \dot{H}^1_{per} \text{ and } v \in L^2([-\pi, \pi])\} \,.$$

(v) Explain the “unitary” part of your answer to (iv) in terms of the energy

$$E(t) = \int_{-\pi}^{\pi} (u_t^2 + u_x^2)\, dx \,.$$

(vi) Show that $\|U(t)\|_{\dot{H}^{s+1}_{per} \oplus \dot{H}^s_{per}} = \|(u_0, u_1)\|_{\dot{H}^{s+1}_{per} \oplus \dot{H}^s_{per}}$ (preservation of regularity).

10. (a) Deduce from the finite speed of propagation result for the wave equation (lemma 5.4.2) that a classical solution of the initial value problem, $\Box u = 0$, $u(0, t) = f$, $u_t(0, x) = g$, with $f, g \in \mathcal{D}(\mathbb{R}^n)$ given is unique.

    (b) The Kirchhoff formula for solutions of the wave equation $n = 3$ for initial data $u(0, \cdot) = 0, u_t(0, \cdot) = g$ is derived using the Fourier transform when $g \in \mathcal{S}(\mathbb{R}^n)$. Show that the validity of the formula can be extended to any smooth function $g \in C^\infty(\mathbb{R}^n)$.(Hint: finite speed of propagation).

11. Write out and prove Stone’s theorem for the case of the finite dimensional Hilbert space $\mathbb{C}^N$ (so that each operator $U(t)$ is now a unitary matrix).