Asymptotic Methods: Example Sheet 1

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The first two questions on background material are not for supervision

1. Based on the definition of the Gamma function in the first lecture, calculate $\Gamma(1/2)$.
2. Calculate $\lim_{R\to+\infty}\int_0^R e^{\pm ix^2}\,dx$, with justification for any change of variables used.

Asymptotic expansions — basic properties

3. Determine whether the sequence of functions $\phi_n(x) = 1-\cosh(x^n)$, for $n\in\{1,2,\dots\}$, is an asymptotic sequence as $x\to 0$.
4. Suppose that the functions f and g have the asymptotic expansions
$$f(z)\sim\sum_{n=0}^{\infty}\frac{a_n}{z^n},\quad\text{and}\quad g(z)\sim\sum_{n=0}^{\infty}\frac{b_n}{z^n},$$
as $z\to\infty$. Show that
$$f(z)g(z)\sim\sum_{n=0}^{\infty}\frac{c_n}{z^n}$$
as $z\to\infty$, where $c_n=\sum_{k=0}^{n}a_{n-k}\,b_k$.
5. Suppose that
$$f(z)\sim a_0+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots,$$
as $z\to\infty$, where $a_0\neq 0$. Show by induction, or otherwise, that
$$\frac{1}{f(z)}\sim\frac{1}{a_0}\sum_{n=0}^{\infty}\frac{d_n}{z^n},$$
as $z\to\infty$, where the sequence $\{d_n\}$ is defined by $d_0=1$ and, for $n\geq 1$,
$$\sum_{k=0}^{n}d_{n-k}\,a_k=0.$$
6. (a) Show that, if a function admits an asymptotic expansion $f(x)\sim\sum_{n=0}^{\infty}a_nx^n$ as $x\to 0^+$, then the a_n are determined uniquely by f.
   (b) Consider the function
   $$e(x)=\exp(-1/x),$$
   for $x>0$. Show that, in an asymptotic expansion of the form
   $$e(x)\sim\beta_0+\beta_1x+\beta_2x^2+\cdots,$$
   valid as $x\to 0^+$, all the coefficients $\beta_0,\beta_1,\beta_2,\dots$ are zero. Deduce that the coefficients $\{a_n\}$ in (a) do not determine f uniquely.

7. (a) Taking δ to be a positive constant, show that as $|z| \to \infty$ in the complex plane (not necessarily along a ray)

$$\cosh z \sim \frac{1}{2}e^z,$$

in the sector $-\pi/2 + \delta < \arg(z) < \pi/2 - \delta$ and

$$\cosh z \sim \frac{1}{2}e^{-z},$$

in the sector $\pi/2 + \delta < \arg(z) < 3\pi/2 - \delta$. Is this still true if $\delta = 0$?

(b) Find asymptotic expansions for $\tanh z$ as $|z| \to \infty$ in the complex plane, stating in which sectors they hold and specifying the Stokes lines.

[Revisit this part at the end of the term when we talk about Stokes lines again.]

8. Show that, if f is continuous and $f(x) = o(\phi(x))$ as $x \to \infty$, where ϕ is a continuous, positive non-decreasing function of x, then

$$\int_a^x f(t)\,dt = o(x\phi(x)),$$

as $x \to +\infty$.

Asymptotic expansions of real integrals

9. (a) Show that the Stieltjes integral,

$$F(x) = \int_0^\infty \frac{\rho(t)}{1+xt}\,dt,$$

admits the asymptotic expansion $F(x) \sim \sum_{n=0}^\infty (-1)^n a_n x^n$ as $x \to 0^+$, where $a_n = \int_0^\infty t^n \rho(t)\,dt$, assuming the continuous function ρ satisfies $0 \le \rho(t) \le C\exp(-\epsilon t)$, for positive C, ϵ and all $t \ge 0$.

If $\rho = xe^{-t}$, show that $F(x) \sim \sum_{n=0}^\infty (-1)^n n!\, x^{n+1}$ as $x \to 0^+$. Similarly, deduce that

$$G(x) = \int_0^\infty \frac{e^{-t}dt}{(1+xt)^2} \sim \sum_{n=0}^\infty (-1)^n (n+1)!\, x^n, \qquad \text{as } x \to 0^+.$$

(b) Differentiating through the integral, show that $F' = G$ and comment on the relation between the two asymptotic series you just obtained.

Give an example of a smooth function, $H : (0,\infty) \to (0,\infty)$, with the property that H admits an asymptotic expansion $\sum \alpha_n x^n$ as $x \to 0^+$, but term-by-term differentiation does not give an asymptotic expansion for H'.

Show, however, that if H' is continuous on $[0,\infty)$, and admits an asymptotic expansion $\sum \beta_n x^n$ as $x \to 0^+$, then necessarily this expansion is given by term-by-term differentiation, i.e. $\beta_n = (n+1)\alpha_{n+1}$.

(c) For a given small positive value of x, find the value(s) of n giving the term(s) of smallest magnitude in the asymptotic expansion for G. Hence, use optimal truncation to obtain an estimate of the exact value $G(0.1) = 0.84366660602\ldots$.

[By convention *optimal truncation* of an asymptotic expansion means keeping all terms in the expansion up to the one BEFORE the smallest.]

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(d) For the case $\rho(t) = e^{-t}$, we find $F(x) = \sum_{n=0}^{N}(-1)^n n! x^n + \mathrm{Err}_N$ with error bound $\mathrm{Err}_N \le (N+1)! x^{N+1}$ (see Section 1 in Chapling's Lecture Notes). Using this to define optimal truncation by $N+1 = [x^{-1}]$, the integer part of x^{-1}, use Stirling's formula to show that the resulting "optimal error bound" is $O\big([x^{-1}]^{1/2}\exp(-[x^{-1}])\big) = o(x^M)$ as $x \to 0^+$, for every positive integer M.

10. (a) Use integration by parts to find an asymptotic expansion, valid as $x \to \infty$, for the *exponential integral*,

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \sim e^{-x}\big(b_1 x^{-1} + b_2 x^{-2} + b_3 x^{-3} + \cdots\big),$$

for suitable constants $b_1, b_2, b_3, \ldots$, showing that the remainder is $O\big(e^{-x}x^{-N-1}\big)$ as $x \to \infty$, for suitable N.

(b) Check your answer by making the substitution $t = x(1+s)$ in the integral and applying Watson's Lemma.

(c) Obtain an asymptotic expansion of $E_1(x)$ *as* $x \to 0^+$ by considering $\frac{d}{dx}[E_1(x) + \ln(x)]$ and integrating.

[You may assume that $\lim_{x\to 0}[E_1(x) + \ln x] = -\gamma$, where γ is Euler's constant.]

11. Find asymptotic expansions of

$$I_1(x) = \int_0^1 e^{-xt(1-t)^2}\,dt \quad \text{and} \quad I_2(x) = \int_0^\infty e^{-xt(1-t)^2}\,dt,$$

as $x \to \infty$, giving all terms up to and including $O(x^{-1})$.

12. By means of Laplace's method, show that the first two terms in an asymptotic expansion of

$$I(x) = \int_0^{\pi/2} \exp\big(-xt^3\cos t\big)\,dt,$$

as $x \to \infty$, are given by

$$I(x) \sim \frac{1}{3x^{1/3}}\Gamma\left(\frac{1}{3}\right) + \left(\frac{1}{6} + \frac{8}{\pi^3}\right)\frac{1}{x} + \cdots.$$

(∗) Find the next term in the expansion.

13. Show that

$$\int_0^{\pi^2/4} \exp\Big(x\cos\sqrt{t}\Big)\,dt \sim e^x\left(\frac{2}{x} + \frac{2}{3x^2} + \cdots\right),$$

as $x \to \infty$.

(∗) Replace the upper limit in the integral with $4\pi^2$ and obtain the first two terms in the corresponding asymptotic expansion.

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