

Asymptotic Methods: Example Sheet 2

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Michaelmas 2023

Two more Integrals of Laplace type

1. Obtain the first correction to the Stirling formula in the asymptotic expansion of the Gamma function, i.e.,

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \dots\right), \quad (x \rightarrow +\infty),$$

2. Derive the expansion:

$$\int_0^{\pi/2} \exp[x(\sin t)^2] dt \sim \frac{e^x}{2} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-m)} \frac{\Gamma(\frac{1}{2}+m)}{m! x^{\frac{1}{2}+m}}, \quad (x \rightarrow +\infty).$$

By means of a change of variables $t \rightarrow \frac{\pi}{2} - t$ and the identity¹

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

or otherwise, obtain the asymptotic expansion

$$\int_0^{\pi/2} e^{-x \sin^2 t} dt \sim \left(\frac{\pi}{4x}\right)^{1/2} \left\{ 1 + \frac{1}{1!} \frac{1^2}{4x} + \frac{1}{2!} \frac{1^2 \cdot 3^2}{(4x)^2} + \dots + \frac{1}{n!} \frac{1^2 \cdot 3^2 \cdot \dots \cdot (2n-1)^2}{(4x)^n} + \dots \right\}.$$

From this obtain an asymptotic expansion, as $x \rightarrow \infty$, for the Bessel function defined by

$$I_0(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos \theta} d\theta.$$

Oscillatory Integrals and Stationary Phase

3. (i) Assume $a < c < b$ and let $f(t)$ be a function which is smooth in $(a, c) \cup (c, b)$ but has a discontinuity at $t = c$. To be precise, assume that for all $n = 0, 1, 2, \dots$ the limits of the n^{th} order derivative $f^{(n)}(t)$ as $t \rightarrow a+, c-, c+$ and $b-$ exist and are designated $f^{(n)}(a+)$, $f^{(n)}(c-)$, $f^{(n)}(c+)$ and $f^{(n)}(b-)$ respectively. Find the asymptotic expansion as $|\omega| \rightarrow \infty$ of

$$I(\omega) = \int_a^b f(t) e^{i\omega t} dt.$$

¹See the course on Further Complex Methods to derive this identity.

(ii) By taking the appropriate limits in part (a), find the asymptotic expansion as $|\omega| \rightarrow \infty$ of $I(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt$, where

$$f(t) = \begin{cases} -e^t & t < 0 \\ e^{-t} & t \geq 0. \end{cases}$$

Compare your result with the exact expression for $I(\omega)$.

4. Review Stokes' problem from the lectures on stationary phase (also see p.22 in Stuart's notes). Obtain the leading asymptotic behaviour as $x \rightarrow \infty$ of

$$\int_a^{\infty} f(t) \exp(ix(t^3 - t)) dt,$$

where f is smooth and $f \rightarrow 0$ as $t \rightarrow \pm \infty$ in the two cases: (i) $a = -\frac{1}{\sqrt{3}}$ and (ii) $a = 1$.

5. Show that, as $x \rightarrow +\infty$,

$$\int_0^{\pi} \exp(ix(t - \sin t)) dt \sim e^{\frac{i\pi}{6}} \left(\frac{6}{x}\right)^{\frac{1}{3}} \Gamma\left(\frac{4}{3}\right).$$

How would this result differ if the lower limit of the integral were $-\pi$?

6. Find the leading term in the asymptotic approximations, valid as $x \rightarrow \infty$, of

(a) $\int_0^1 \cos(xt^p) dt$, with $p > 1$, real,

(b) $\int_0^{\frac{\pi}{2}} \left(1 - \left(\frac{2\theta}{\pi}\right)\right)^{\gamma} \cos(x \cos \theta) d\theta$, for $\gamma = 0$, $\gamma = -\frac{1}{2}$ and $\gamma = -\frac{3}{4}$.

Method of Steepest Descent

7. The function $f(\theta)$ is defined for θ real and positive by

$$f(\theta) = \frac{1}{2\pi i} \int_{\gamma} \exp\left(\theta\left(t + \frac{1}{3}t^3\right)\right) dt,$$

where the path γ , in the complex plane, begins at ∞ in the sector $-\frac{\pi}{2} < \arg t < -\frac{\pi}{6}$ and ends at ∞ in the sector $\frac{\pi}{6} < \arg t < \frac{\pi}{2}$. Find the two saddle points and show that the two paths of steepest descent through these points are

$$x = + \left((2 + y)/3y \right)^{\frac{1}{2}} (y - 1), \quad y > 0$$

and

$$x = - \left((y - 2)/3y \right)^{\frac{1}{2}} (y + 1), \quad y < 0,$$

where $t = x + iy$. You should justify carefully your choice of signs for the square roots. Show that, as $\theta \rightarrow \infty$,

$$f(\theta) = (\pi\theta)^{-\frac{1}{2}} \cos\left(\frac{2\theta}{3} - \frac{\pi}{4}\right) + O(\theta^{-1}).$$

8. Use the method of steepest descent to obtain the first two non-zero terms in the asymptotic approximation

$$\int_0^\infty \exp\left(ix\left(\frac{1}{3}t^3 + t\right)\right) dt \sim i\left(\frac{1}{x} + \frac{2}{x^3} + \dots \frac{a_n}{x^n} + \dots\right),$$

as $x \rightarrow +\infty$. Check your answer by doing an integration by parts/stationary phase argument to the integral as it stands.

(*) Find an expression for a_n for all n .

9. Let

$$h(t) = i(t + t^2).$$

Sketch the path through the point $t = 0$ for which $\text{Im}(h(t)) = \text{const}$. Sketch also the path through the point $t = 1$ for which $\text{Im}(h(t)) = \text{const}$.

By integrating along these paths, show that, as $\lambda \rightarrow \infty$,

$$\int_0^1 t^{-\frac{1}{2}} \exp\left(i\lambda(t + t^2)\right) dt \sim \frac{c_1}{\lambda^{\frac{1}{2}}} + c_2 \frac{e^{2i\lambda}}{\lambda} + \dots,$$

where the constants c_1 and c_2 are to be determined.

10. (*) Apply the method of steepest descent to the integral

$$I(k) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\exp[k(z - 2z^{1/2})]}{z - c} dz,$$

for the case $k \rightarrow +\infty$. Here the path of integration is parallel to the imaginary axis, and $\gamma > 1$ is a real constant. The branch cut for \sqrt{z} is the negative real axis. Show that the two parameterized curves $\tau \rightarrow z_\pm(\tau)$ given by

$$z_\pm(\tau) = 1 - \tau^2 \pm 2i\tau, \quad 0 \leq \tau < \infty,$$

are the steepest descent paths emanating from the saddle-point $z = 1$, and show that they form two halves of a parabola crossing the real axis at the saddle point; find the equation of the parabola in real form.

Investigate the asymptotics of $I(k)$ as $k \rightarrow +\infty$ in the following cases:

- (i) c is real and < 1 ;
- (ii) c is real, $1 < c < \gamma$;
- (iii) $c = ib$ with b real and $b > 2$.

[You may find it convenient to use τ as a variable of integration.]