Asymptotic Methods: Example Sheet 3

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One more Steepest Descent example

1. The Hankel function $H_\nu^{(1)}(x)$ is defined by

$$H_\nu^{(1)}(x) = -\frac{1}{\pi}\int_{-i\infty}^{i\infty-\pi} e^{i(\nu t - x\sin t)}\,dt,$$

where the path of integration, in the complex plane, satisfies $-\pi \le \mathrm{Re}(t) < 0$. Use the method of steepest descent to show that, as $\nu \to +\infty$, one has

$$H_\nu^{(1)}\Big(\frac{\nu}{\cos\alpha}\Big) \sim \sqrt{2}\,(\pi\nu\tan\alpha)^{-1/2} e^{i\nu(\tan\alpha-\alpha)} e^{-i\pi/4},$$

where $0 < \alpha < \pi/2$. (Thus x/ν remains a positive constant greater than 1, as $\nu \to \infty$.) Find an equation for the path along which you integrate, sketch it and justify that it is indeed the path of steepest descent.

Liouville–Green and WKBJ approximations

2. (a) Consider the differential equation,

$$zw'' - zw' - w = 0,$$

in a neighbourhood of the point at infinity, $z = \infty$. Classify this point.

(b) Show that one solution to the differential equation is $w = ze^z$. Hence find an expression for a second independent solution as an integral.

(c) Find $\mu(z)$ and $q(z)$ such that $W(z) = w(z)/\mu(z)$ satisfies $W'' - q(z)W = 0$. Write down expressions for the two independent Liouville–Green solutions of this equation in terms of $q(z)$.

Calculate the first two terms of the expansion of $\sqrt{q(z)}$ about $z = \infty$ in inverse powers of z, and hence determine the leading-order asymptotic form for w as $z \to \infty$ from the Liouville–Green solutions. Compare these with the two exact solutions obtained previously.

3. (a) Given $\varepsilon^2 y'' = q(x)y$, consider solutions of the form

$$y = \exp\Big(\frac{1}{\varepsilon}S_0 + S_1 + \varepsilon S_2 + \varepsilon^2 S_3 + \cdots\Big),$$

for small ε. Show that the functions $\{S_n\}_{n=0}^\infty$ are determined iteratively by

$$(S_0')^2 = q, \quad 2S_0'S_1' + S_0'' = 0,$$

and

$$2S_0'S_n' + S_{n-1}'' + \sum_{j=1}^{n-1} S_j'S_{n-j}' = 0.$$

Find expressions for S_0, S_1, and S_2 in terms of q (taking care with the sign choice in S_0).

(b) Apply the above solution method to the Airy equation $y'' = xy$, as follows: let[1] $\varepsilon = 1$, $q(x) = x$ and find a relation which determines $S_n(x)$ in terms of the functions $\{S_j(x)\}_{j=1}^{n-1}$. Show inductively that this defines a sequence of functions of the form $S_j(x) = c_j x^{(3/2)(1-j)}$ for $j \geq 2$ and some constant c_j. Deduce that Ai might reasonably be expected to admit an asymptotic expansion of the form[2]

$$\mathrm{Ai}(x) \sim Ax^{-1/4} \exp\left(-\frac{2}{3}x^{3/2}\right)\left(1 - \frac{5}{48}x^{-3/2} + O(x^{-3})\right),$$

as $x \to \infty$ and where A is a constant.

4. Show that, for large positive x, the equation,

$$w'' - x^3 w' + x^{-2} w = 0,$$

possesses two independent solutions with asymptotic behaviour

$$w \sim 1 + O(x^{-4}) \quad \text{and} \quad w \sim x^{-3} \exp\left(\frac{x^4}{4}\right)\left(1 + O(x^{-4})\right).$$

5. The wavefunction ψ obeys the equation,

$$-\frac{d^2\psi}{dx^2} - \frac{s(s+1)}{\cosh^2(x)}\psi = E\psi,$$

with $s > 0$. Use the WKBJ approximation to find approximate values for the bound-state energies (eigenvalues E for which $\psi \to 0$ as $|x| \to \infty$),

$$\sqrt{-E} = \sqrt{s(s+1)} - n - \frac{1}{2}.$$

What are the allowed values of n? Compare your eigenvalues with the exact bound-state energies $E = -(s-n)^2$, where $n \in \{0, 1, 2, \dots\}$ and $n < s$.

[*Hint:* You may find the substitution $\sinh x = z$ useful, and note that if a, b are real,

$$\int_{-a}^{a} \frac{\sqrt{a^2 - z^2}}{b^2 + z^2}\, dz = \pi\left(\sqrt{1 + \frac{a^2}{b^2}} - 1\right),$$

which can be derived via contour integration.]

[1] One way to justify the procedure of introducing the small parameter ε and then setting it to one is via the rescaling $X = \varepsilon^{2/3}x$, assuming that $\varepsilon^{2/3}x = \mathcal{O}(1)$ as $x \to \infty$. One then derives the asymptotic relations for the $\{S_j(X)\}$ using part (a), notices that they are in fact *scale invariant*, and then rescales back.

[2] You are not expected to prove that this procedure gives a complete asymptotic expansion. The integral formulation for the Airy functions provides an efficient basis for a rigorous approach to their asymptotics. See Section 7.6 in Hormander's "Analysis of Linear Partial Differential Operators I" or the appendix to "Complex Analysis" by Stein and Shakarchi for some proofs.

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6. Show that the large eigenvalues of the system

$$\frac{d^2w}{dx^2} = (x^4 + x^2 - \lambda^2)w, \quad w(-\infty) = w(\infty) = 0,$$

are given by

$$\lambda \sim 2^{1/3}\pi\left(\Gamma\left(\frac{1}{4}\right)\right)^{-4/3}\left(3n + \frac{3}{2}\right)^{2/3} + O(1),$$

where n is a (large) positive integer.

7. $(*)$ Quantum tunnelling through a potential barrier in the stationary approach to scattering theory is described by the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi,$$

where $V \geq 0$ and $V \to 0$ faster than $1/|x|$ as $|x| \to \infty$. Consider a particle incident from $x = -\infty$ with an energy $E = \hbar^2k^2/(2m)$, and assume that $E < V$ on an interval (a, b) and $E > V$ on $(-\infty, a)$ and (b, ∞). This means that for $x \to -\infty$ the wave function should approach $e^{ikx} + re^{-ikx}$ while for $x \to +\infty$ the wave function should approach te^{ikx}, where the constants r and t are known as the reflection and transmission coefficients of the potential. (The term e^{ikx} corresponds to the incident wave, travelling to the right from $x = -\infty$ in the corresponding time-dependent description.)

Write down Liouville-Green approximations to the wave functions in the three regions (a, b), $(-\infty, a)$ and (b, ∞), consistent with the setup just described and indicating clearly which are oscillatory and which are real exponential solutions. Show using the connection formulae of the WKBJ approximation that the transmission probability $|t|^2$ is approximately

$$\exp\left(-\frac{2}{\hbar}\int_a^b \sqrt{2m(V(x) - E)}\, dx\right),$$

when the integral is large compared with $\hbar$.

[*Hint:* You will need to include both $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ when matching across $x = a$ and $x = b$. Refer to Section VII.2 in Stuart's notes or pages 57—58 in Chapling's notes. For a careful treatment of this problem see Landau and Lifshitz, Quantum Mechanics, 3rd ed., p. 178, or Bender and Orszag p. 524.]

Further examples of Stokes' phenomenon and the asymptotics of integrals

8. The analytic function $f(z)$ is defined by

$$f(z) = \int_{i\infty}^{z} \exp(s^2)\, ds.$$

Use the method of integration by parts to obtain an asymptotic expansion of the form

$$f(z) \sim \exp(z^2)\left(\frac{1}{2z} + \frac{1}{4z^3} + \frac{3}{8z^5} + \frac{15}{16z^7} + \cdots\right),$$

in the sector $0 \leq \arg(z) \leq \pi$, including an expression for the remainder after a finite number of term and a verification of the asymptotic property.

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To deal with the case $-\pi \le \arg(z) \le 0$, convert $f(z)$ to the form

$$f(z) = \int_{-i\infty}^{z} \exp(s^2)\, ds - i\pi^{1/2},$$

and, without making detailed calculations, obtain an asymptotic expansion valid in this domain for $f(z) + i\pi^{1/2}$. Find the Stokes lines for these asymptotic expansions of f as $|z| \to \infty$.

9. $(*)$ Stuart's notes considers the methods of stationary phase and steepest descent as applied to

$$I(z) = \int_0^1 \exp(zt^3)\, dt,$$

when $z = ix$ is purely imaginary (see Sections V.4 and VI). This question is concerned with the general case $z \in \mathbb{C}$.

(a) When $\mathrm{Re}(z) > 0$, show that

$$I(z) + \frac{\Gamma(4/3)}{z^{1/3}} \sim \frac{e^z}{3} \sum_{n=0}^{\infty} \frac{\Gamma(n+2/3)}{\Gamma(2/3)} \frac{1}{z^{n+1}},$$

as $|z| \to \infty$, in the sector $\arg(z) \in (-\pi/2+\delta, \pi/2-\delta)$ for small positive δ, and obtain a similar relation for the sector $\arg(z) \in (\pi/2+\delta, 3\pi/2-\delta)$.

[*Hint:* Use the complex version of Watson's lemma; see Section IV.4 in Stuart's notes.]

(b) By generalizing the application of the method of steepest descent in Stuart's notes, or otherwise, develop the asymptotic expansion,

$$I(z) \sim \frac{e^z}{3} \sum_{n=0}^{\infty} \frac{\Gamma(n+2/3)}{\Gamma(2/3)} \frac{1}{z^{n+1}} + \frac{e^{i\pi/3}\Gamma(4/3)}{z^{1/3}},$$

as $|z| \to \infty$, in the sector $\arg(z) \in (\delta, \pi-\delta)$. Find a similar relation for the sector $\arg(z) \in (\pi+\delta, 2\pi-\delta)$. Identify any Stokes lines. Discuss the different expansions of (a) and (b) in their regions of overlap.

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