1.1. Let g^t be the flow map associated with the smooth vector field V, so that $\boldsymbol{x}(t) = g^t\boldsymbol{x}_0$ is the unique solution to the ODE $\dot{\boldsymbol{x}} = V(\boldsymbol{x})$ with $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Show that:

$$g^0 = I, \quad g^{t+s} = g^t g^s, \quad g^{-t} = (g^t)^{-1}.$$

1.2. Let ψ_1^s and ψ_2^s be *commuting* 1-parameter groups of transformations generated by the smooth vector fields V_1 and V_2 respectively. Show that $\psi^s = \psi_1^s \circ \psi_2^s$ also defines a 1-parameter group of transformations and show that it is generated by $V = V_1 + V_2$.

Conversely, show that if a 1-parameter group of transformations ψ^s is generated by $V = V_1 + V_2$ where $[V_1, V_2] = 0$, then $\psi^s = \psi_1^s \circ \psi_2^s$ where the ψ_1^s and ψ_2^s are generated by V_1 and V_2 as before.

1.3. Write both of the following 1-parameter groups of transformations as a composition of commutative 1-parameter transformations:

$$\psi_1^s(x_1, x_2, x_3) = (x_1 + s, x_2 + 2s, x_3 + 3s), \quad \psi_2^s(x_1, x_2, x_3) = (e^s x_1, e^{2s} x_2, e^{3s} x_3)$$

Hence write down the vector fields which generate these transformations,and check your answers are correct by showing the relevant ODEs are satisfied.

1.4. Establish the Leibniz rule (derivation property) and the Jacobi identity of Poisson brackets:

$$\{f, gh\} = \{f, g\}\, h + \{f, h\}\, g, \qquad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

Deduce that if $f, g : M \to \mathbb{R}$ are two first integrals of the Hamiltonian system (M, H), then so is $h = \{f, g\}$.

1.5. Consider as in the discussion of canonical transformations the linear maps $U : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $U^T J U = J$, where J is as in the Hamilton equations. Show that these form a group, and also that J is itself symplectic and that U is symplectic iff U^T is symplectic. (For the first part you may find it convenient to consider the symplectic form $\omega(X, Y) = X^T J Y$.)

1.6. Let $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p})$ and $\boldsymbol{y} = (\boldsymbol{Q}, \boldsymbol{P})$. Using the chain rule, show that a smooth coordinate transformation $\boldsymbol{x} \mapsto \boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$ whose derivative[1] $D\boldsymbol{y}(\boldsymbol{x})$ is symplectic preserves the form of Hamilton's equations $\dot{\boldsymbol{x}} = J\nabla H(\boldsymbol{x})$ for some transformed Hamiltonian, which you should give. Give an example of a linear transformation for which $D\boldsymbol{y}$ is not symplectic but which preserves the form of Hamilton's equations, and give the transformed Hamiltonian. (Hint: scale.) By expressing $D\boldsymbol{y}(\boldsymbol{x})$ in terms of $\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{Q}, \boldsymbol{P}$, show that $\boldsymbol{x} \mapsto \boldsymbol{y}(\boldsymbol{x})$ has symplectic derivative if and only if:

$$\{Q_i, Q_j\}_{q,p} = \{P_i, P_j\}_{q,p} = 0, \qquad \{Q_i, P_j\}_{q,p} = -\{P_j, Q_i\}_{q,p} = \delta_{ij}$$

Please send any corrections to dmas2@cam.ac.uk

Questions marked $(*)$ are optional and should not be attempted at the expense of unstarred questions

[1]Here $D\boldsymbol{y}$ denotes the Jacobian matrix with entries $(D\boldsymbol{y})_{ij} = \partial y_i / \partial x_j$.

1.7. Consider the four dimensional phase space with coordinates $(\boldsymbol{q}, \boldsymbol{p}) = (\phi, r, p_\phi, p_r)$ and take as Hamiltonian:

$$H(\phi, r, p_\phi, p_r) = \frac{p_\phi^2}{2r^2} + \frac{p_r^2}{2} - \frac{\alpha}{r}$$

where α is a positive constant. Use the fact that $\partial_\phi H = 0$ to show the existence of two first integrals in involution and deduce that the system is integrable in the sense of the Arnol'd-Liouville theorem. Show that on the level set $M_{\boldsymbol{c}} = \{H = c_1, p_\phi = c_2\}$ the coordinate p_r can be written in the form:

$$p_r^2 = 2c_1 + \frac{2\alpha}{r} - \frac{c_2^2}{r^2} \equiv -\frac{2c_1}{r^2}(r_1 - r)(r - r_2),$$

where you should find r_1, r_2 explicitly. The coordinates $(\phi . p_\phi) = (\phi, I_\phi)$ already look like an "action-angle" pair. Construct the remaining action-angle coordinates by considering

$$I_r = \frac{1}{2\pi} \oint_{\Gamma_r} \boldsymbol{p} \cdot d\boldsymbol{q},$$

where Γ_r is the cycle on $M_{\boldsymbol{c}}$ on which $\phi = \text{const}$. Conclude that

$$H(\phi, r, p_\phi, p_r) = \tilde{H}(I_\phi, I_r) = -\frac{\alpha^2}{2(|I_\phi| + I_r)^2}.$$

1.8. Let $L(t)$ and $A(t)$ be $n \times n$ matrices depending differentiably on $t \in \mathbb{R}$, and such that

$$\frac{dL}{dt} = [L, A]. \tag{1}$$

Show, without considering the eigenvalues/vectors of L, that $\mathrm{tr}(L^p)$, $p \in \mathbb{N}$, does not depend on t.

1.9. Show that if in (1) the matrix A is skew-symmetric ($A^T = -A$) and L is symmetric then both sides of the equation are symmetric.

1.10. Write down the Hamiltonian equations for the Toda Hamiltonian for N particles moving in one dimension, $H = \frac{1}{2}\sum_{j=1}^N p_j^2 + \sum_{j=1}^{N-1} \exp(q_j - q_{j+1})$ and show that with the definitions $a_j = \frac{1}{2}\exp[(q_j - q_{j+1})/2]$ and $b_j = -\frac{1}{2}p_j$ they are equivalent to

$$\dot{a}_j = a_j(b_{j+1} - b_j)\,, \qquad \dot{b}_j = 2(a_j^2 - a_{j-1}^2)\,. \tag{2}$$

(Use the convention that $q_0 = -\infty, e^{q_0} = 0, q_{N+1} = +\infty, e^{-q_{N+1}} = 0$.)

1.11. Recall the Toda problem with $N = 2$ can be written as the Lax pair $\dot{L} = [B, L]$ with

$$L = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}.$$

Express the eigenvalues of L in terms of the total momentum $p_1 + p_2$ and the energy H, check they are in involution. Obtain the general solution to the system.

1.12. Extend the Lax pair formulation of the Toda problem to general N, by considering the tri-diagonal[2] $N \times N$ matrices whose diagonal elements are $L_{jj} = b_j$ and $B_{jj} = 0$ for $j = 1, \dots N$, and whose near diagonal elements are $L_{j,j+1} = L_{j+1,j} = a_j$ and $B_{j,j+1} = -B_{j+1,j} = a_j$ for $j = 1, \dots N-1$. Show that the equations (2) are equivalent to $\dot{L} = [B, L]$. For the case $N = 3$ deduce that $F_1 = \lambda_1 + \lambda_2 + \lambda_3$, $F_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$ and $F_3 = \lambda_1\lambda_2\lambda_3$ are all 1st integrals (where λ_j are the eigenvalues of L, which you may assume to be real and distinct). Calculate F_1, F_2, F_3 in terms of $a_1, a_2, b_1 b_2, b_3$ and hence show that F_1, F_2, F_3 are in involution. * Prove the eigenvalues of L are real and distinct.

1.13 (*)**.** Let g^t be the flow associated with the Hamiltonian vector field $X_H = J\nabla H$. If $\boldsymbol{x}(0) = \boldsymbol{y}$, use Taylor's theorem to show that:

$$g^t \boldsymbol{y} = \boldsymbol{y} + t X_H(\boldsymbol{y} + o(t). \qquad (\star)$$

Let $D(t) = g^t D(0)$ be a region in M evolving via the Hamiltonian flow and let $\mathrm{Vol}(t)$ denote the volume of the region. By making the change of variables $\boldsymbol{x}(t) = g^t \boldsymbol{y}$, where $\boldsymbol{y} \in D(0)$, show that:

$$\mathrm{Vol}(t) \equiv \int_{D(t)} d^{2m}\boldsymbol{x} = \int_{D(0)} \det\left(\frac{\partial x_i}{\partial y_j}\right) d^{2m}\boldsymbol{y}.$$

Using $(\star)$ and $\det(I + \epsilon A + o(\epsilon)) = 1 + \epsilon \mathrm{tr} A + o(\epsilon)$ for any matrix A, deduce that the derivative of $\mathrm{Vol}(t)$ vanishes at $t = 0$. What is the value of the derivative at an arbitrary $t = t_0$? Deduce that the Hamiltonian flow preserves volume. This is known as Liouville's theorem and is an important result in statistical mechanics and ergodic theory.

[2] A tri-diagonal matrix is one whose only nonzero elements are either on the diagonal or nearest neighbour to the diagonal.