

2.0. Given the linear differential operators

$$L = -\partial_x^2 + u(x, t), \quad A = 4\partial_x^3 - 3u\partial_x - 3\partial_x u,$$

considered as acting on a vector space of functions of  $x$  verify the fact from lectures that the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0$$

is equivalent to  $L_t = [L, A]$ .

2.1. Consider a family of Hermitian  $N \times N$  matrices  $L(t)$  obeying

$$L(t) = U(t)L(0)U(t)^\dagger$$

where  $U(t)$  is unitary. Show that  $L(t)$  and  $L(0)$  have the same eigenvalues. Assuming differentiability, show that there exists a skew-Hermitian matrix  $A$  (i.e., a matrix obeying  $A = -A^\dagger$ ) such that  $U_t = -AU$  and  $L_t = [L, A]$ . Give a condition on the family of matrices  $\{U(t)\}$  which implies  $A$  is independent of time.

Conversely, suppose  $L$  satisfies  $L_t = [L, A]$  for some skew-Hermitian  $A$ . Let  $U(t)$  be the solution of  $U_t = -AU$ ,  $U(0) = I$ . Show that  $U(t)$  is unitary. By considering  $\frac{d}{dt}U(t)^\dagger L(t)U(t)$  or otherwise, show that  $L(t) = U(t)L(0)U(t)^\dagger$ . (Do not assume that  $A$  is independent of time.)

2.2. Suppose  $u = u(x, t)$  satisfies a Hamiltonian evolution equation  $u_t = \mathcal{J}\delta H$ . Show that if  $I = I[u]$  then  $I_t = \{I, H\}$ , where  $\{F, G\} = \langle \delta F, \mathcal{J}\delta G \rangle$ . Assuming that this is a Poisson bracket on the space of functionals which satisfies the conditions for a Poisson structure in Ib of the notes, deduce that if  $I_1$  and  $I_2$  are conserved, then so is  $I_3 = \{I_1, I_2\}$ .

2.3. Show that KdV  $u_t + u_{xxx} - 6uu_x = 0$  can be written in Hamiltonian form in two distinct ways

$$H_0[u] = \int \frac{1}{2}u^2 dx, \quad \mathcal{J}_0 = -\partial_x^3 + 4u\partial_x + 2u_x \quad \text{and} \quad H_1[u] = \int \left(\frac{1}{2}u_x^2 + u^3\right) dx, \quad \mathcal{J}_1 = \partial_x.$$

In both cases check that the operator  $\mathcal{J}$  is anti-symmetric.

2.4. Let  $\xi = x - ct$ . By seeking a solution of the form  $u(x, t) = f(\xi)$ , find the 1-soliton solution to each of the following nonlinear (integrable) PDEs:

$$\begin{aligned} \text{KdV:} \quad & u_t + u_{xxx} - 6uu_x = 0, \\ \text{Sine-Gordon:} \quad & u_{tt} - u_{xx} + \sin u = 0. \end{aligned}$$

In the first case, assume that  $f(\xi)$  approaches zero as  $|\xi| \rightarrow \infty$ , and in the second case assume it approaches values  $f_\pm$  as  $\xi \rightarrow \pm\infty$  which are consistent with finite energy:  $\int \frac{u_t^2 + u_x^2}{2} + (1 - \cos u) dx < \infty$ . In each case you will find it useful to multiply an ODE by  $f'(\xi)$ . Plot the solutions for two differing values of time  $t$ .

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**2.5.** Let  $L = -\frac{d^2}{dx^2} + u(x)$  be the one dimensional Schrödinger operator with potential  $u$ , assumed to decay rapidly at infinity. Show that if  $L\psi = \lambda\psi$  and  $L\psi' = \lambda\psi'$  then the Wronskian  $W(\psi, \psi') \equiv \psi\psi'_x - \psi'\psi_x$  is constant. Using this fact establish the following results concerning the discrete and continuous parts of the spectrum of  $L$  respectively:

- (i) Show that if  $\psi$  and  $\psi'$  are bound states corresponding to the same discrete eigenvalue then  $\psi \propto \psi'$ . Deduce that the discrete eigenvalues are non-degenerate, i.e. each discrete eigenvalue corresponds to exactly one bound state.
- (ii) Show that the reflection and transmission coefficients obey  $|R(k)|^2 + |T(k)|^2 = 1$  for all  $k$ . [Hint: if  $L\Phi = k^2\Phi$  then  $L\bar{\Phi} = k^2\bar{\Phi}$  also, the bar denoting complex conjugation.]

**2.6.** Referring to the operators  $L, A$  defining the Lax structure of KdV in Exercise 0, show that  $L$  is selfadjoint and  $A$  is skew-adjoint:  $\langle \varphi, L\psi \rangle = \langle L\varphi, \psi \rangle$ ,  $\langle \varphi, A\psi \rangle = -\langle A\varphi, \psi \rangle$  for any smooth, rapidly decaying functions  $\psi$  and  $\varphi$ . If  $\psi$  is a real function with  $\|\psi(t)\| = 1 \forall t$  and  $\tilde{\psi}(t) = \psi_t(t) + A\psi(t)$ , show that  $\psi$  and  $\tilde{\psi}$  are orthogonal, i.e.  $\langle \tilde{\psi}, \psi \rangle = 0$ . Conclude that if  $u$  satisfies the KdV equation and  $\psi$  is a bound state for  $L$  then  $\psi_t + A\psi = 0$  and obtain the time dependence of the discrete part of the scattering data associated to the potential  $u$ . [Hint: use question 2.5(i)].

**2.7.** Recall from lectures that if  $A = \partial_x + m \tanh mx$  and  $A^\dagger = -\partial_x + m \tanh mx$ , then

$$-\partial_x^2 + m^2 = AA^\dagger \quad \text{and} \quad -\partial_x^2 + m^2 - 2m^2 \operatorname{sech}^2 mx = A^\dagger A, \quad (1)$$

from which we found the scattering data for the potential  $-2m^2 \operatorname{sech}^2 mx$  was  $R(k) = 0$  and  $\chi_1^2 = -m^2$  and  $c_1 = \sqrt{2}m$ . Now by considering  $B = \partial_x + 2m \tanh mx$  and  $B^\dagger = -\partial_x + 2m \tanh mx$ , and computing  $BB^\dagger$  and  $B^\dagger B$ , find the scattering data for the potential  $-6m^2 \operatorname{sech}^2 mx$ . (Hint: consider  $B^\dagger A^\dagger e^{ikx} = (-k^2 - 3imk \tanh mx + 2m^2 - 3m^2 \operatorname{sech}^2 mx) e^{ikx}$  to find the reflection coefficient, and argue similarly to the case  $-2m^2 \operatorname{sech}^2 mx$  treated in lectures for the bound state scattering data.)

(\*) Verify that solving Gelfand-Levitan-Marcenko equation for  $K = K(x, y)$  and defining  $u(x) = -2\frac{d}{dx}K(x, x)$  leads back to  $u = -6m^2 \operatorname{sech}^2 mx$ .

**2.8.** The  $N = 2$  soliton solution to the KdV is given by ( $\chi_1 > \chi_2$ )

$$u(x, t) = -8 \left[ \frac{(\chi_1^2 e^{\eta_1} + \chi_2^2 e^{\eta_2}) + 2(\chi_1 - \chi_2)^2 e^{\eta_1 + \eta_2} + \alpha_{12}(\chi_1^2 e^{\eta_1 + 2\eta_2} + \chi_2^2 e^{2\eta_1 + \eta_2})}{(1 + e^{\eta_1} + e^{\eta_2} + \alpha_{12} e^{\eta_1 + \eta_2})^2} \right]$$

where  $\eta_i(x, t) = 2\chi_i x - 8\chi_i^3 t + \beta_i$  for  $i = 1, 2$  and  $\alpha_{12} = (\chi_1 - \chi_2)^2 (\chi_1 + \chi_2)^{-2}$ . By setting  $\eta_1 = \text{const}$  and taking the limit  $t \rightarrow \infty$  show that in a frame of reference travelling at speed  $4\chi_1^2$  the 2-soliton reduces to a one soliton solution

$$u(x, t) = -2\chi_1^2 \operatorname{sech}^2[\chi_1(x - 4\chi_1^2 t) + \phi_\infty]$$

where you should determine the constant  $\phi_\infty$ . By instead taking the limit  $t \rightarrow -\infty$ , calculate the phase shift  $\Delta\phi = \phi_\infty - \phi_{-\infty}$  induced by the soliton interaction.

**2.9.** (i) Given  $u = u(x, t) \in \mathbb{R}$  define

$$U(\lambda) = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix} \quad \text{and} \quad V(\lambda) = \begin{pmatrix} -u_x & 2u + 4\lambda \\ 2u^2 - u_{xx} + 2u\lambda - 4\lambda^2 & u_x \end{pmatrix}. \quad (2)$$

Show that the KdV equation is equivalent to the zero curvature equation

$$U_t - V_x + [U, V] = 0. \quad (3)$$

(ii) Let  $v = v(x, t)$  be a complex valued function. Show that the matrices  $(U, V)$  defined by

$$U(\lambda) = \begin{pmatrix} i\lambda & i\bar{v} \\ iv & -i\lambda \end{pmatrix}, \quad V(\lambda) = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & \bar{v} \\ v & 0 \end{pmatrix} + \begin{pmatrix} 0 & \bar{v}_x \\ -v_x & 0 \end{pmatrix} - i \begin{pmatrix} |v|^2 & 0 \\ 0 & -|v|^2 \end{pmatrix}$$

satisfy the zero curvature equation iff  $v$  satisfies the *nonlinear Schrödinger* (NLS) equation

$$iv_t + v_{xx} + 2|v|^2v = 0.$$

This is another integrable equation which arises in optics and the mathematical theory of water waves.

**2.10.** This exercise asks you to compute some detailed consequences of the formulae in section III.e of the notes, in which it is explained how to use the zero curvature formulation of sine-Gordon to “solve” it. In particular, it is possible to recover multi-soliton solutions of  $u_{XT} = \sin u$  at time  $T$  from certain discrete scattering data  $\{\lambda_m(T), c_m(T)\}_{m=1}^N$  and corresponding eigenfunctions  $\psi_m(X, T)$  solving by means of the formula

$$u_X(X, T) = -4 \sum_m c_m \psi_m^{(1)}(X, T) e^{i\lambda_m X}$$

where  $\psi_m = \begin{pmatrix} \psi_m^{(1)} \\ \psi_m^{(2)} \end{pmatrix}$  and  $\tilde{\psi}_m = \begin{pmatrix} -\overline{\psi_m^{(2)}} \\ \overline{\psi_m^{(1)}} \end{pmatrix}$  solve

$$\tilde{\psi}_n(X, T) e^{i\bar{\lambda}_n(T)X} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sum_m \frac{c_m(T) \psi_m(X, T)}{(\bar{\lambda}_n(T) - \lambda_m(T))} e^{i\lambda_m(T)X}. \quad (4)$$

Given the fact that the discrete scattering data  $\{\lambda_m(T), c_m(T)\}_{m=1}^N$  evolve according to  $\lambda_m(T) = \lambda_m(0) = \lambda_m$  and  $c_m(T) = c_m(0) e^{-\frac{i}{2\lambda_n} T}$ , obtain the solution in the case that  $N = 2$  and  $\lambda_j = il_j$ ,  $j = 1, 2$  with *real*  $l_2 > l_1 > 0$  and for arbitrary *real*  $c_1(0), c_2(0)$ .

Changing coordinates to  $x = X + T$  and  $t = T - X$  explain briefly how, for large positive and negative  $t$ , your formula approximates to a combination of the 1-solitons given by

$$4 \arctan \exp\left[-\frac{x - v_i t - x_{i0}}{\sqrt{1 - v_i^2}}\right]$$

with velocities  $v_1$  and  $v_2$ ; calculate these velocities in terms of  $l_1, l_2$ .