3.1. Let v be any solution of the wave equation in double-null coordinates: $v_{xt} = 0$. Show that the two equations:

$$u_x + v_x = \sqrt{2}\exp\left(\frac{u-v}{2}\right), \qquad u_t - v_t = \sqrt{2}\exp\left(\frac{u+v}{2}\right),$$

are compatible iff u satisfies Liouville's equation $u_{xt} = e^u$. These equations constitute a Bäcklund transformation. By considering the most general form of $v = v(x,t)$, show that:

$$u(x,t) = 2\log\left(-\frac{\sqrt{2}}{\int^x \exp[-f(\xi)]d\xi + \int^t \exp[g(\tau)]d\tau}\right) + g(t) - f(x).$$

3.2. Using the notation of qu. 3 on sheet II, show that if there exists a sequence $\{H_n\}_{n=0}^{\infty}$ of functionals satisfying

$$\mathcal{J}_1\frac{\delta H_{n+1}}{\delta u} = \mathcal{J}_0\frac{\delta H_n}{\delta u}\,,$$

then these are all 1st integrals (conserved quantities) for KdV.

3.3. i) Find the vector fields V_1, V_2, V_3 which generate the following smooth one-parameter groups of transformations of $\mathbb{R}$:

$$x \mapsto \psi_1^s x = x + s, \quad x \mapsto \psi_2^s x = e^s x, \quad x \mapsto \psi_3^s x = \frac{x}{1 - sx}.$$

ii) Deduce that these vector fields generate a group of transformations of the form

$$x \mapsto \frac{ax+b}{cx+d}, \quad ad - bc = 1.$$

iii) Compute the structure constants $\{f_{ij}^k\}_{i,j,k=1}^3$ defined by $[V_i, V_j] = \sum_{k=1}^3 f_{ij}^k V_k$.

iv) (*) Show that these transformations can be understood as arising from a smooth left action of $SL(2) = \{A \in \mathrm{mat}(2\times 2) | \det A = 1\}$ on $\mathbb{R}$.

3.4. Compute the 1-parameter group of transformations generated by

$$V = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Find new coordinates (X, Y) with $X = X(x,y)$ and $Y = Y(x,y)$ such that $V(X) = 1$ and $V(Y) = 0$. Use your results to integrate the ODE

$$x^2\frac{dy}{dx} = F(xy),$$

where F is an arbitrary function of one variable.

Please send any corrections to dmas2@cam.ac.uk

Questions marked (∗) are optional and should not be attempted at the expense of unstarred questions

3.5. Write each of the following 1-parameter groups of transformations as a composition of commutative 1-parameter transformations:

$$\psi_1^s(x,t,u) = (x{+}s, t{+}2s, u{+}3s), \quad \psi_2^s(x,t,u) = (e^s x, e^s t, u{-}s), \quad \psi_3^s(x,t,u) = (e^s x, t{+}us, u).$$

Hence *write down* the vector fields which generate these transformations. Check your answers are correct by showing the relevant ODEs are satisfied. Show that

$$\psi^s(x,t,u) = (x\cosh s + t\sinh s, x\sinh s + t\cosh s, u)$$

defines a 1-parameter group of transformations. Does the previous method fail in this case? Find the generator of ψ^s and comment on the aforementioned failure.

3.6. Let $\tilde{\mathbf{x}} = \psi^s \mathbf{x}$ be a new set of coordinates where ψ^s is a 1-parameter group of transformations with generator V. Use Taylor's theorem to show (formally) that for nice functions f

$$f(\tilde{\mathbf{x}}) = f(\mathbf{x}) + sVf(\mathbf{x}) + \frac{s^2}{2!}V(Vf)(\mathbf{x}) + \frac{s^3}{3!}V(V(Vf))(\mathbf{x}) + \cdots = \sum_{n=0}^{\infty}\frac{s^n}{n!}(V^n f)(\mathbf{x}).$$

Deduce that, at least formally, $\psi^s \equiv \exp(sV)$. Show that $\exp(s\partial_x)x = x + s$ and $\exp(sx\partial_x)x = e^s x$.

3.7. Let ψ^s be a 1-parameter group of transformations generated by V. A function $F = F(\mathbf{x})$ is said to be an *invariant* of ψ^s if $F(\psi^s\mathbf{x}) = F(\mathbf{x})$ for all $\mathbf{x}$. Show that F is an invariant if and only if $VF(\mathbf{x}) = 0$.

3.8. Compute the 1-parameter groups of transformations associated with the vector fields

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{\partial}{\partial u} + \alpha t\frac{\partial}{\partial x}, \quad V_4 = \beta x\frac{\partial}{\partial x} + \gamma t\frac{\partial}{\partial t} + \delta u\frac{\partial}{\partial u}.$$

Find the constants $(\alpha, \beta, \gamma, \delta)$ for which these vector fields generate symmetries of the KdV equation. Determine the structure constants in the corresponding 4-dimensional Lie algebra of vector fields.

3.9. Let $u = u(x)$. Calculate the first prolongation of the following 1-parameter groups of transformations

$$\psi_1^s(x,u) = (x{+}s, u), \quad \psi_2^s(x,u) = (e^s x, u{+}s), \quad \psi_3^s(x,u) = (x\cos s{-}u\sin s, x\sin s{+}u\cos s).$$

Let V_1, V_2, V_3 be the corresponding generators. Using your answers to the previous part, show that

$$\mathrm{pr}^{(1)}\, V_1 = V_1, \quad \mathrm{pr}^{(1)}\, V_2 = V_2 - u_x\frac{\partial}{\partial u_x}, \quad \mathrm{pr}^{(1)} V_3 = V_3 + (1+u_x^2)\frac{\partial}{\partial u_x}.$$

Without looking at your notes, derive the first prolongation formula and verify these are correct.

3.10. Let $u = u(x,t)$. The vector field $V = \xi\partial_x + \phi\partial_t + \eta\partial_u$ generates a 1-parameter group of transformations

$$(x,t,u) \mapsto (\tilde{x},\tilde{t},\tilde{u}) = (x + s\xi(x,t,u), t + s\phi(x,t,u), u + s\eta(x,t,u)) + o(s).$$

By considering the contact condition $d\tilde{u} = \tilde{u}_{\tilde{t}}d\tilde{t} + \tilde{u}_{\tilde{x}}d\tilde{x}$ show that $\mathrm{pr}^{(1)}V = V + \eta^x\partial_{u_x} + \eta^t\partial_{u_t}$ where

$$\eta^t = \mathrm{D}_t\eta - u_t\mathrm{D}_t\phi - u_x\mathrm{D}_t\xi, \quad \eta^x = \mathrm{D}_x\eta - u_x\mathrm{D}_x\xi - u_t\mathrm{D}_x\phi,$$

where D_x and D_t are total derivatives.

3.11. The modified KdV equation is $v_t + v_{xxx} - 6v^2v_x = 0$. Find a Lie-point symmetry of the form

$$\psi^s(x,t,v) = (e^{\alpha s}x, e^{\beta s}t, e^{\gamma s}v)$$

for appropriate numbers (α,β,γ). Consider the group invariant solution $v(x,t) = (3t)^{-1/3}w(z)$, where $z = x(3t)^{-1/3}$, and construct a 3rd order differential equation for w. Integrate this equation once to show that w satisfies Painlevé II.

3.12 (*)**.** Let $u = u(x)$ and $V = \xi\partial_x + \eta\partial_u$. Calculate $\mathrm{pr}^{(2)}V$. Show that the equation $u_{xx} = 0$ admits an 8 dimensional group of Lie-point symmetries. Can you give geometrical meaning to each of the generators?