Mathematical Tripos Part II **Michaelmas Term 2026**
Principles of Quantum Mechanics **Dr. J.M. Evans**

Example Sheet 1

1. A Hamiltonian is defined by

$$H = \sum_n \Big(\mathcal{E}\,|n\rangle\langle n| \,+\, \lambda\,|n{+}1\rangle\langle n| \,+\, \lambda\,|n\rangle\langle n{+}1| \,\Big)$$

where n runs over all integers (positive and negative) and labels a basis of orthonormal states satisfying $\langle n|n'\rangle = \delta_{nn'}$, and $\mathcal{E}$ and λ are constants. Show that the states defined by $|\theta\rangle = \sum_n e^{in\theta}|n\rangle$ are eigenstates of H and determine the corresponding energy eigenvalues.

2. Write down the position $\hat{x}$ and momentum $\hat{p}$ of a one-dimensional harmonic oscillator in terms of the annihilation and creation operators a and $a^\dagger$. Use $[a, a^\dagger] = 1$ to evaluate $[a, a^{\dagger\,n}]$.

If $|0\rangle$ is a normalized state such that $a|0\rangle = 0$, verify that $|\chi\rangle = a^{\dagger\,n}|0\rangle$ is an eigenstate of $N = a^\dagger a$ and find its norm.

Calculate the expectation values of $\hat{x}$, $\hat{p}$, $\hat{x}^2$ and $\hat{p}^2$ in the state $|\chi\rangle$ and show that $\Delta x \Delta p = (n + \frac{1}{2})\hbar$.

3. Let A and B be any operators which each commute with their commutator $[A, B]$.

(a) Prove that

$$[A, B^n] = nB^{n-1}[A, B] \;\; (n = 1, 2, \ldots) \qquad \text{and} \qquad [A, e^B] = e^B[A, B]\,.$$

(b) Let $F(\lambda) = e^{\lambda A}e^{\lambda B}e^{-\lambda(A+B)}$, an operator-valued function of the parameter λ. Show that $F'(\lambda) = \lambda[A, B]F(\lambda)$ and hence deduce that

$$e^A e^B \;=\; e^{A+B+\frac{1}{2}[A,B]} \;=\; e^B e^A e^{[A,B]}\;.$$

4. Prove that, for any operators A and B,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(e^{\lambda A}Be^{-\lambda A}) \;=\; e^{\lambda A}[A, B]e^{-\lambda A}\,.$$

By repeated use of this relation, find a series expansion in λ for $e^{\lambda A}Be^{-\lambda A}$ and deduce that

$$e^A Be^{-A} \;=\; B \,+\, [A, B] \,+\, \tfrac{1}{2}[A, [A, B]] \,+\, \ldots\;.$$

5. Let $X(t) = e^{iHt/\hbar}\,\hat{x}\,e^{-iHt/\hbar}$ and $P(t) = e^{iHt/\hbar}\,\hat{p}\,e^{-iHt/\hbar}$ where $\hat{x}$ and $\hat{p}$ are the usual position and momentum operators and $H(\hat{x}, \hat{p})$ is the Hamiltonian for a harmonic oscillator of mass m and frequency ω in one dimension. Show that

$$\begin{aligned} X(t) \;&=\; \hat{x}\,\cos(\omega t) \,+\, \tfrac{1}{m\omega}\,\hat{p}\,\sin(\omega t)\,, \\ P(t) \;&=\; \hat{p}\,\cos(\omega t) \,-\, m\omega\,\hat{x}\,\sin(\omega t)\,. \end{aligned}$$

Evaluate $[X(t), P(t)]$ and $H(X(t), P(t))$.

6. The operators b and $b^\dagger$ satisfy the relations

$$b\,b^\dagger + b^\dagger b = 1 \quad \text{and} \quad b^2 = b^{\dagger\,2} = 0\,.$$

Show that the eigenvalues of $N = b^\dagger b$ can only be 0 or 1. Assuming the existence of a normalized state $|0\rangle$ with eigenvalue 0, construct a normalized state $|1\rangle$ with eigenvalue 1, and show how $|0\rangle$ can be expressed in terms of $|1\rangle$. Obtain the matrix elements $\langle n|b|n'\rangle$ and $\langle n|b^\dagger|n'\rangle$ (for $n, n' = 0, 1$) and check that these matrices obey the correct algebra (i.e. the relations for the operators given above).

7. Consider a harmonic oscillator with standard annihilation and creation operators a and $a^\dagger$ and ground state $|0\rangle$. For any complex number α, a *coherent state* is defined by

$$|\alpha\rangle \;=\; e^{\alpha a^\dagger - \alpha^* a}|0\rangle \;=\; C_\alpha\, e^{\alpha a^\dagger}|0\rangle\,.$$

Check that the second equality holds for a certain constant C_α, to be determined. Show that $|\alpha\rangle$ is an eigenstate of a and find its eigenvalue. Calculate the inner-product $\langle\alpha|\beta\rangle$ of any two coherent states. Find the expectation values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ in a general coherent state $|\alpha\rangle$.

8. Show that in position space the annihilation operator for the simple harmonic oscillator has the representation

$$a = \frac{1}{\sqrt{2}}\Big(\frac{\mathrm{d}}{\mathrm{d}y} + y\Big) \qquad \text{where} \qquad y = \Big(\frac{m\omega}{\hbar}\Big)^{\frac{1}{2}} x\,.$$

Use the relation $a|0\rangle = 0$ to calculate the wave function $\psi_0(y)$ of the ground state. Use the position space representation of the creation operator to show that the wave function for the n-th excited state is given by

$$\psi_n(y) = h_n(y)\,\psi_0(y) \qquad \text{where} \qquad h_n(y) = \frac{(-1)^n}{(2^n n!)^{\frac{1}{2}}}\, e^{y^2}\frac{\mathrm{d}^n}{\mathrm{d}y^n}e^{-y^2} \quad \text{is a polynomial of degree } n\,.$$

9. Let $|\psi\rangle$ be any state of a system in three dimensions with $|\mathbf{x}\rangle$ and $|\mathbf{p}\rangle$ eigenstates of position and momentum, satisfying $\langle\mathbf{x}|\mathbf{p}\rangle = e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}/(2\pi\hbar)^{3/2}$.

(a) Show that for $L_z = \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x$,

$$\langle\mathbf{x}|L_z|\psi\rangle \;=\; -i\hbar\Big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\Big)\psi(\mathbf{x}) \qquad \text{and} \qquad \langle\mathbf{p}|L_z|\psi\rangle \;=\; -i\hbar\Big(p_x\frac{\partial}{\partial p_y} - p_y\frac{\partial}{\partial p_x}\Big)\tilde{\psi}(\mathbf{p})$$

where $\psi(\mathbf{x})$ and $\tilde{\psi}(\mathbf{p})$ are the position space and momentum space wavefunctions for $|\psi\rangle$.

(b) Find the position space wavefunction of $e^{-i\mathbf{a}\cdot\hat{\mathbf{p}}/\hbar}|\psi\rangle$ and the momentum space wavefunction of $e^{i\mathbf{k}\cdot\hat{\mathbf{x}}}|\psi\rangle$ where $\mathbf{a}$ and $\mathbf{k}$ are constant vectors.

10. Obtain the Schrödinger equation for the hydrogen atom in the momentum space representation. Show that $\tilde{\psi}(\mathbf{p}) = N/(\mathbf{p}^2+\alpha^2)^2$ is a solution for a suitable choice of the constant α and find its energy eigenvalue. The following integrals may be used:

$$\frac{4\pi}{\mathbf{k}^2} = \int \mathrm{d}^3x\; e^{-i\mathbf{k}\cdot\mathbf{x}}\;\frac{1}{|\mathbf{x}|}\,, \qquad \int \mathrm{d}^3p'\;\frac{1}{(\mathbf{p}-\mathbf{p}')^2}\;\frac{1}{(\mathbf{p}'^2+\alpha^2)^2} = \frac{\pi^2}{\alpha(\mathbf{p}^2+\alpha^2)}\,.$$

11. Determine the energy eigenvalues of the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \tfrac{1}{2}m\omega^2(\hat{x}-c)^2\,,$$

by setting $\hat{x} = c + (\hbar/2m\omega)^{\frac{1}{2}}(a+a^\dagger)$ and $\hat{p} = (\tfrac{1}{2}\hbar m\omega)^{\frac{1}{2}} i(a^\dagger - a)$. Why are the energies independent of the constant c?

The Hamiltonian for a particle of charge e in a constant magnetic field B along the z direction is

$$H = \frac{1}{2m}(\hat{\mathbf{p}} - e\mathbf{A}(\hat{\mathbf{x}}))^2\,, \qquad \text{with} \qquad A_x = A_z = 0\,, \quad A_y = B\hat{x}\;.$$

Show that $\hat{p}_y, \hat{p}_z, H$ are a set of commuting operators. Consider states on which $\hat{p}_y, \hat{p}_z$ take the definite values p_y, p_z and show that the possible energy eigenvalues are

$$E = \frac{p_z^2}{2m} + (n+\tfrac{1}{2})\hbar\omega\,, \qquad \omega = \frac{|eB|}{m}\,,$$

independent of p_y.

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