

Principles of Quantum Mechanics - Problems 2

Please *email me* with any comments about these problems, particularly if you spot an error. Problems with an asterisk (*) are optional and may be more difficult.

1. Let \mathbf{n} be the unit vector in the direction with polar coordinates (θ, ϕ) and let $\boldsymbol{\sigma}$ be the Pauli matrices. Find the eigenvectors of $\mathbf{n} \cdot \boldsymbol{\sigma}$. Hence show that the state of a spin- $\frac{1}{2}$ particle in which a measurement of the component of spin along \mathbf{n} is certain to yield $\hbar/2$ is

$$|\uparrow_{\mathbf{n}}\rangle = \sin \frac{\theta}{2} e^{i\phi/2} |\downarrow\rangle + \cos \frac{\theta}{2} e^{-i\phi/2} |\uparrow\rangle. \quad (*)$$

where $|\uparrow\rangle, |\downarrow\rangle$ are the usual eigenstates of S_z . Obtain the corresponding expression for $|\downarrow_{\mathbf{n}}\rangle$. Explain why $\langle \uparrow | \uparrow_{\mathbf{n}} \rangle = 0$ at $\theta = \pi$, and why each of the coefficients in (*) has modulus $1/\sqrt{2}$ when $\theta = \pi/2$.

2. For a spin- $\frac{1}{2}$ particle, the spin operator $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$, where $\boldsymbol{\sigma}$ is the vector $(\sigma_1, \sigma_2, \sigma_3)$ each entry of which is a Pauli matrix.
- i) Without multiplying matrices, explain why the Pauli matrices obey

$$\sigma_i \sigma_j = \delta_{ij} 1_{2 \times 2} + i\epsilon_{ijk} \sigma_k$$

for $i, j, k \in \{1, 2, 3\}$. Hence show that $\mathbf{a} \cdot \boldsymbol{\sigma} \mathbf{b} \cdot \boldsymbol{\sigma} = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}$ for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

- ii) Hence show that $e^{-i\boldsymbol{\alpha} \cdot \mathbf{S}/\hbar} = \cos\left(\frac{\alpha}{2}\right) \text{id} - i \sin\left(\frac{\alpha}{2}\right) \hat{\boldsymbol{\alpha}} \cdot \boldsymbol{\sigma}$ for a spin- $\frac{1}{2}$ particle.
- iii) A particle of spin- $\frac{1}{2}$ interacts with a uniform, homogeneous magnetic field \mathbf{B} via the Hamiltonian $H = -\gamma \mathbf{B} \cdot \mathbf{S}$, where γ is a constant. If the particle's spin is initially prepared to be in some state $|\chi\rangle$, show that the probability its spin is found to be in an orthogonal state $|\chi'\rangle$ a time t later is $|\langle \chi' | \hat{\mathbf{B}} \cdot \boldsymbol{\sigma} | \chi \rangle|^2 \sin^2 \omega t$, where ω is a frequency you should specify.
- iv) Obtain the Heisenberg equation of motion for $\mathbf{S}(t)$ with this Hamiltonian.
- v) Hence show that the spin operator in the Heisenberg picture is

$$\mathbf{S}(t) = \cos 2\omega t \mathbf{S} + (1 - \cos 2\omega t) \hat{\mathbf{B}} (\hat{\mathbf{B}} \cdot \mathbf{S}) - \sin 2\omega t \hat{\mathbf{B}} \times \mathbf{S}.$$

3. A spin- $\frac{1}{2}$ particle interacts with a time-varying magnetic field such that

$$H = -\gamma \mathbf{B}(t) \cdot \mathbf{S} \quad \text{where} \quad \mathbf{B}(t) = B_0 \hat{\mathbf{z}} + b(\hat{\mathbf{x}} \cos \omega_1 t + \hat{\mathbf{y}} \sin \omega_1 t)$$

Let $|\psi(t)\rangle$ be the state of the particle at time t and let $U(\omega_1 t \hat{\mathbf{z}})$ be the rotation operator around the $\hat{\mathbf{z}}$ -axis through the time dependent angle $\omega_1 t$.

- i) Define $|\chi(t)\rangle$ by $|\psi(t)\rangle = U(\omega_1 t \hat{\mathbf{z}}) |\chi(t)\rangle$. Show that $|\chi(t)\rangle$ obeys the TDSE with Hamiltonian $H_{\text{eff}} = -\gamma \mathbf{B}_{\text{eff}} \cdot \mathbf{S}$ where \mathbf{B}_{eff} is a time-independent magnetic field you should specify.
- ii) Hence show that $\langle \mathbf{S} \rangle_{\psi(t)} = \mathbf{R}(\omega_1 t \hat{\mathbf{z}}) \mathbf{R}(\gamma t \mathbf{B}_{\text{eff}}) \langle \mathbf{S} \rangle_{\psi(0)}$, where $\mathbf{R}(\boldsymbol{\alpha})$ is a rotation matrix in \mathbb{R}^3
- iii) Draw a sketch to illustrate how $\langle \mathbf{S} \rangle_{\psi(t)}$ varies in time.

4. Write down the 3×3 matrix that represents S_x for a spin-1 system in the basis in which $S_z = \text{diag}(\hbar, 0, -\hbar)$.

An unpolarized beam of spin-1 particles enters a Stern–Gerlach filter that passes only particles with $S_z = \hbar$. On exiting this filter, the beam enters a second filter that passes only particles with $S_x = \hbar$ and then finally it encounters a filter that passes only particles with $S_z = -\hbar$. What fraction of the initial particles make it right through?

5. Let $R = |\mathbf{X}|$ and $P_r = (\hat{\mathbf{X}} \cdot \mathbf{P} + \mathbf{P} \cdot \hat{\mathbf{X}})/2$ be operators corresponding to the radial coordinate and momenta. Evaluate $[R, P_r]$.

The radial Hamiltonian of a Hydrogen atom in a state of definite total angular momentum labelled by ℓ is

$$H_\ell = \frac{P_r^2}{2\mu} + V_{\text{eff}}(R) = \frac{\hbar^2}{2\mu} \left(\frac{P_r^2}{\hbar^2} + \frac{\ell(\ell+1)}{R^2} - \frac{2}{a_0 R} \right),$$

where μ is the reduced mass and $a_0 = 4\pi\epsilon_0\hbar^2/\mu e^2$ is the Bohr radius.

- i) Sketch the effective potential for various $\ell \in \mathbb{N}_0$. By considering their minima, show that for any fixed $E < 0$ there is a maximum value of ℓ beyond which bound states of energy E cannot exist.
- ii) Let A_ℓ be the dimensionless operator

$$A_\ell = \frac{a_0}{\sqrt{2}} \left(\frac{1}{(\ell+1)a_0} - \frac{\ell+1}{R} + \frac{i}{\hbar} P_r \right)$$

Express H_ℓ in terms of A_ℓ and its Hermitian adjoint, and show that

$$[A_\ell, A_\ell^\dagger] = \frac{a_0^2 \mu}{\hbar^2} (H_{\ell+1} - H_\ell).$$

Hence show that $A_\ell H_\ell = H_{\ell+1} A_\ell$.

- iii) Let $|E, \ell\rangle$ obey $H_\ell |E, \ell\rangle = E |E, \ell\rangle$. Show that $A_\ell |E, \ell\rangle$ is an eigenstate of $H_{\ell+1}$ with the same eigenvalue E . What is the interpretation of the radial wavefunction $\langle r | A_\ell |E, \ell\rangle$?
- iv) Explain why there must exist an $\ell_{\text{max}} \in \mathbb{N}_0$ s.t. $A_{\ell_{\text{max}}} |E, \ell_{\text{max}}\rangle = 0$. Deduce that $|E, \ell_{\text{max}}\rangle$ has energy

$$E = -\frac{\hbar^2}{2\mu a_0^2} \frac{1}{(\ell_{\text{max}} + 1)^2}$$

and hence that the energy levels of Hydrogen are

$$E = -\frac{\hbar^2}{2\mu a_0^2} \frac{1}{n^2}$$

with orbital angular momentum $\ell \in \{0, 1, \dots, n-1\}$.

- 6*. Consider a $d = 2$ isotropic harmonic oscillator of frequency ω .

- i) Construct combinations of the raising and lowering operators of the $2d$ oscillator that obey the same $\mathfrak{so}(3)$ algebra as angular momentum in $3d$.
- ii) Show how all the states of the $2d$ oscillator fit into representations of $\mathfrak{so}(3)$.