Mathematical Tripos Part II **Michaelmas Term 2026**
Principles of Quantum Mechanics **Dr. J.M. Evans**

Example Sheet 2

1. Starting from the Schrödinger equation for states, or from the Heisenberg equation for operators, verify that

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi|A|\psi\rangle = \langle\phi|[A,H]|\psi\rangle$$

holds in either the Schrödinger picture or the Heisenberg picture.

2. Consider a harmonic oscillator initially in a state $|\psi(0)\rangle = |\alpha\rangle$, where $|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle = C_\alpha\, e^{\alpha a^\dagger}|0\rangle$ is a normalised coherent state, as in question 7 on example sheet 1. By applying the time evolution operator in the Schrödinger picture, show that the state at a later time t is $|\psi(t)\rangle = e^{-i\omega t/2}|e^{-i\omega t}\alpha\rangle$. Hence find expressions for

$$\langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle \quad \text{and} \quad \langle\psi(t)|\,\hat{p}\,|\psi(t)\rangle\ .$$

Now refer to the results of question 5 on example sheet 1 and note that $X(t)$ and $P(t)$ are, in fact, the position and momentum operators in the Heisenberg picture. Express these operators in terms of a and $a^\dagger$ (defined from $\hat{x}$ and $\hat{p}$) and hence find directly $\langle\alpha|\,X(t)\,|\alpha\rangle$ and $\langle\alpha|\,P(t)\,|\alpha\rangle$.

3. The Hamiltonian for a point particle of mass m and charge e in a constant electric field $\mathbf{E}$ is

$$H(\hat{\mathbf{x}},\hat{\mathbf{p}}) = \frac{1}{2m}\hat{\mathbf{p}}^2 - e\,\mathbf{E}\cdot\hat{\mathbf{x}}\ .$$

Working in the Heisenberg picture, solve the Heisenberg equations of motion for $\hat{\mathbf{x}}(t)$ and $\hat{\mathbf{p}}(t)$, expressing your answer in terms of $\hat{\mathbf{x}}(0)$ and $\hat{\mathbf{p}}(0)$. Verify that $H(\hat{\mathbf{x}}(t),\hat{\mathbf{p}}(t)) = H(\hat{\mathbf{x}}(0),\hat{\mathbf{p}}(0))$.

4. Consider an isotropic harmonic oscillator in three dimensions with mass m and potential

$$V(\hat{\mathbf{x}}) = \frac{1}{2}m\omega^2(\hat{x}_1^2+\hat{x}_2^2+\hat{x}_3^2)\ .$$

Express the Hamiltonian H in terms of annihilation and creation operators a_i and $a_i^\dagger$ for three one-dimensional oscillators ($i=1,2,3$). Explain how to construct the space of states on which these operators act and hence determine the energy levels for H. What is the degeneracy of the n-th energy level?

5. In a simple model of the atomic nucleus, each constituent particle, or nucleon, is a proton or neutron moving in a harmonic potential $V(\mathbf{x})$, as in 4 above; this potential represents the interaction of any given nucleon with all the others. Explain why this model suggests that the nuclear isotopes ^{4}He, ^{16}O and ^{40}Ca should be especially stable. Here the superscripts indicate the total number of nucleons for these nuclei, which contain 2, 8 and 20 protons, respectively. [Recall that protons and neutrons are particles with spin-$\frac{1}{2}$; you may find it helpful to review arguments relating to the structure of atoms which suggest that certain chemical elements should be unreactive.]

6. (a) The trace of an operator A acting on a space of states V is defined as $\mathrm{Tr}_V(A) = \sum_n \langle n|A|n\rangle$ where $\{|n\rangle\}$ is an orthonormal basis for V. Show that this definition is independent of the choice of basis.

Suppose that $V = V_1 \otimes V_2$ and $A = A_1 \otimes A_2$ where A_1, A_2 act on V_1, V_2 respectively. Show that $\mathrm{Tr}_V(A) = \mathrm{Tr}_{V_1}(A_1)\,\mathrm{Tr}_{V_2}(A_2)$.

(b) If $a, a^\dagger$ are annihilation, creation operators and V is the usual space of states for an oscillator, show that $\mathrm{Tr}(x^{a^\dagger a}) = (1-x)^{-1}$ for $|x|<1$.

***** Now let $a_i, a_i^\dagger$ be a set of D annihilation, creation operators obeying $[a_i, a_j^\dagger] = \delta_{ij}$. Show that

$$\mathrm{Tr}(x^{\mathcal{N}}) = (1-x)^{-D} \qquad \text{where} \qquad \mathcal{N} = \sum_{i=1}^{D} a_i^\dagger a_i\ .$$

Find the degeneracy d_n of the eigenvalue n of the operator $\mathcal{N}$, by first explaining why

$$\mathrm{Tr}(x^{\mathcal{N}}) = \sum_{n\geq 0} d_n x^n.$$

7. A particle of mass m is confined to a three-dimensional box with sides $a < b < c$ (the wavefunction for the particle vanishes on the boundary of the box). Show that, with a suitable choice of coordinates, the allowed eigenfunctions and energy levels are

$$\psi_{npq}(x,y,z) = \Big(\frac{8}{abc}\Big)^{\frac{1}{2}} \sin\frac{n\pi x}{a}\sin\frac{p\pi y}{b}\sin\frac{q\pi z}{c}\ , \qquad E_{npq} = \frac{\hbar^2\pi^2}{2m}\Big(\frac{n^2}{a^2}+\frac{p^2}{b^2}+\frac{q^2}{c^2}\Big)$$

where $n, p, q = 1, 2, \ldots$. [You may quote results for a particle confined to a one-dimensional box.]

What are the two lowest energy levels, their degeneracies and associated wave functions, for two identical non-interacting spin-0 particles confined to the box? What are the corresponding results for two non-interacting, identical, spin-$\frac{1}{2}$ particles?

8. A one dimensional oscillator of mass m and frequency ω is perturbed by a term $\lambda\hat{x}^4$. Show that the change in energy of each state to first order in perturbation theory is

$$\delta E_n = 3\lambda(2n^2+2n+1)\Big(\frac{\hbar}{2m\omega}\Big)^2 \qquad n = 0, 1, 2, \ldots .$$

***** For what values of λ would you expect perturbation theory to converge? [Hint: consider the behaviour of the entire potential when x is large.]

9. The quantum-mechanical observable Q has just three eigenstates, $|1\rangle,\ |2\rangle,\ |3\rangle$ which are orthogonal and correspond to eigenvalues 2, 1, 1 respectively. The operator Q' is defined by $Q' = Q + \epsilon T$, where

$$\langle j|T|i\rangle = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & \sqrt{6} \\ 0 & \sqrt{6} & 1 \end{pmatrix} .$$

Quoting the necessary formulae (or deriving them if you prefer), apply perturbation theory to find two of the eigenvalues of Q' correct to order ϵ and the third correct to order ϵ^2.

10. Two harmonic oscillators of frequency ω and mass m interact through a potential $m\omega^2\lambda x_1 x_2$ where x_1 and x_2 are the displacements of the oscillators.

(i) Write down a basis of energy eigenstates for the system when $\lambda = 0$ (no interaction). What are the energy levels and what is the degeneracy of each level?

(ii) Use perturbation theory to calculate the ground state energy of the interacting system, correct to second order in powers of λ.

(iii) Find the energies of the first two excited states, working to lowest non-trivial order in λ.

(iv) Verify your results by using the change of variables $\sqrt{2}\,q_1 = x_1 + x_2,\ \sqrt{2}\,q_2 = x_1 - x_2$ to obtain an exact expression for the energy eigenvalues.

(v) For what values of λ do the arguments in (iv) hold, and what can you deduce about the perturbation expansion used in (ii) and (iii)?

11. The first excited energy level of the hydrogen atom is four-fold degenerate (ignoring electron spin). The four wave functions may be taken to be

$$\psi_0(\mathbf{x}) = \frac{1}{(8\pi a^3)^{\frac{1}{2}}}\Big(1-\frac{r}{2a}\Big)\,e^{-r/2a}\,, \quad \psi_i(\mathbf{x}) = \frac{1}{(32\pi a^5)^{\frac{1}{2}}}x_i\,e^{-r/2a}\,, \qquad a = \frac{4\pi\epsilon_0\hbar^2}{e^2 m}\,,$$

where x_i are the Cartesian components of the vector $\mathbf{x}$. The atom is perturbed by a weak electric field $\mathcal{E}$ parallel to the 3-axis. Show that the only non zero matrix elements of the perturbation H' are $\langle\psi_0|H'|\psi_3\rangle$ and $\langle\psi_3|H'|\psi_0\rangle$. Calculate the new energy levels to first order in $\mathcal{E}$.

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