1. A classical, non-perfect gas of $N$ atoms has Hamiltonian

$$H = \sum_r \frac{1}{2m} |p_r|^2 + \sum_{r<s} U(R_{rs})$$

where $R_{rs} = |x_r - x_s|$. Show that the partition function is $Z = Z_0 K$ where $Z_0$ is the partition function for a perfect gas and

$$K = \frac{1}{V^N} \int \exp \left[ -\frac{1}{T} \sum_{r<s} U(R_{rs}) \right] \prod_r d^3 x_r.$$

Let $\lambda_{rs} = 1 - \exp[-U(R_{rs})/T]$ and work to first order in the quantities $\lambda_{rs}$. Show that $K \approx 1 - \frac{N^2}{V} B(T)$ where $B(T) = \frac{1}{2} \int [1 - e^{-U(y)/T}] d^3 y$, $y = |y|$. Deduce that $P \approx \frac{N^2}{V} [1 + \frac{N}{V} B(T)]$ (so $B(T)$ is the second virial coefficient). Assuming that $U$ is everywhere positive or everywhere negative, comment on the relative signs of $U$ and $B$.

2. For a non-perfect gas as in Q.1, suppose the atoms repel each other according to $U(y) = \alpha/y^n$ where $n > 3$ and $\alpha > 0$. Find $B(T)$. Evaluate the special case $n = 6$. (For general $n$, the numerical factor in the integral reduces to a gamma function, but it simplifies for $n = 6$.)

3. * Exercise to show that the canonical equilibrium distribution function in classical phase space depends only on energy, $E$.

Let phase space have coordinates $(q_i, p_i)$, $1 \leq i \leq 3N$. Consider $\rho$, the probability density in phase space, as the density of a “gas” or “ensemble” of points in phase space, each moving according to Hamilton’s equations, with Hamiltonian $H(q_i, p_i)$. Let $v$ be the velocity of the “gas”, with components $(\dot{q}_i, \dot{p}_i)$. Show that

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v)$$

where $\nabla$ is the divergence operator in $6N$ dimensions. Use Hamilton’s equations to deduce that

$$\frac{\partial \rho}{\partial t} = -\sum_i \left( \frac{\partial \rho}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

and show that this vanishes if $\rho$ is a function of $H$. Conversely, show that if $\rho$ is time-independent, and Hamilton’s equations are satisfied, then $\rho$ is constant along the trajectories of points in phase space, and deduce that if the energy is effectively the only conserved quantity then $\rho$ is a function of $H$, and therefore a function of the conserved energy $E$. 
4. Compute the equation of state, including the second virial coefficient, for a gas of non-interacting hard discs of radius \( \frac{r_0}{2} \) in two dimensions. [For hard discs, the potential is infinite if the discs overlap, and zero otherwise.]

5. A classical gas of atoms in three dimensions is constrained by a wall to move in the \( x \geq 0 \) region of space. A potential

\[
V(x) = \frac{1}{2} \alpha x^2
\]

attracts the atoms to the wall. The atoms are free to move in an area \( A \) in the \( y \) and \( z \) directions. If the gas is at uniform temperature \( T \), show that the number density of atoms varies as

\[
\rho(x) = 2N_0 \sqrt{\frac{\alpha}{2\pi T}} e^{-\alpha x^2/2T}
\]

where \( N_0 \) is the total number of atoms per unit area. By considering the balance of forces on a slab of gas between \( x \) and \( x + \Delta x \), show that locally the gas continues to obey the ideal gas law. Hence determine the pressure that the gas exerts on the wall.

6. A particle moving in one dimension has Hamiltonian

\[
H = \frac{p^2}{2m} + \lambda q^4
\]

Show that the heat capacity for a gas of \( N \) such particles is \( C = \frac{3}{4} N \). Explain why the heat capacity is the same regardless of whether the particles are distinguishable or indistinguishable.

7. Consider a set of quantum harmonic oscillators with a range of angular frequencies \( \omega_j \) at temperature \( T \), and discard the zero point energy, so the energy levels of each oscillator are \( \varepsilon_n = n\hbar \omega_j \) (Planck oscillators). Show that this system is thermodynamically equivalent to a bosonic particle system in contact with a heat and particle reservoir, in which there are 1-particle energy levels \( \hbar \omega_j \) that can be occupied by any number of particles \( n \). What is the temperature and chemical potential of the reservoir? [Assume the thermodynamic equivalence applies to each oscillator.]

8. Treat the rotational degrees of freedom of a gas of \( N \) diatomic molecules by quantum theory as follows. Assume that the rotational state of a molecule is determined by two quantum numbers \( J, M \), which are integers satisfying \( |M| \leq J \); and that the energy is \( \varepsilon_{J,M} = \frac{\hbar^2}{2I} (J + 1) \), where \( I \) is the moment of inertia. Find the partition function

\[
Z_{\text{rot}} = \left( \sum_{J,M} \exp(-\varepsilon_{J,M}/T) \right)^N
\]

by approximating the sum over \( J \) as an integral. Find the average value of the rotational energy.

9. For \( n \) a positive integer, let

\[
I_n = \int_0^\infty \frac{x^n}{e^x - 1} \, dx.
\]
By expanding \((e^x-1)^{-1}\) as \(e^{-x} + e^{-2x} + e^{-3x} + \cdots\), show that \(I_n = n! \zeta(n+1)\), where 
\[ 
\zeta(n) = \sum_{r=1}^{\infty} r^{-n} \]
is the Riemann zeta function. Recall the value \(\zeta(4) = \frac{\pi^4}{90}\) (see e.g. Riemann Zeta Function of 4 - ProofWiki), and hence obtain the value of \(I_3\).

10. Consider a plane surface bounding a region in which there is black-body radiation with energy density \(e(\omega)d\omega\) in frequency range \((\omega, \omega + d\omega)\). Show that the energy per unit time incident upon unit area of the surface is \(\frac{1}{3}ce(\omega)d\omega\).

[Hint: remember that the speed of a photon is \(c\). Consider photons incident at an angle \(\theta\) to the normal to the surface, and integrate over solid angle.]

11. Consider blackbody radiation at temperature \(T\). Show that the average number of photons grows as \(T^3\). What is the mean photon energy? What is the most likely energy of a photon?

12. A black body at temperature \(T\) absorbs all the radiation that falls on it and emits radiation at the rate \(E = \sigma T^4\) per unit area, where \(\sigma\) is Stefan’s constant. A black, perfectly conducting sphere orbits a star of radius \(7 \times 10^5\) km at a distance of \(1.5 \times 10^8\) km. The star radiates like a black body at temperature 6000 K. Can you make a gin and tonic on this sphere?