Electrodynamics

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Lecture notes

Anthony Challinor
a.d.challinor@ast.cam.ac.uk

1 Introduction

Electrodynamics is the most successful field theory in theoretical physics and it has provided a model for all later developments. This course develops from IB *Electromagnetism*. We start in Part I by developing electromagnetism as a classical relativistic field theory, showing how the relativistic form of Maxwell's equations, introduced in the IB course, can be derived from a variational principle, and presenting the covariant treatment of the energy and momentum carried by the electromagnetic field. Such a treatment is essential for later use in quantum field theory. The remainder of the course shows how Maxwell's equations describe realistic phenomena. Part II discusses the production of electromagnetic radiation by accelerated charges, and mechanisms for scattering radiation are briefly introduced. Finally, Part III treats electromagnetism in continuous media, which allows one to understand some of the tremendous diversity in the electric and magnetic properties of materials. The propagation of electromagnetic waves in media is a particular focus.

In this course, we assume familiarity with special relativity as met in the IA course *Dynamics and Relativity* and in the IB course *Electromagnetism*. Appendix A provides a fairly detailed recap if you feel you need it. You may want to skim this anyway as it establishes our conventions and notations, which may not be the same as in your earlier courses.

Part I

Electromagnetism as a classical field theory

2 Review of Maxwell's equations in relativistic form

Maxwell's equations in their usual 3D form are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday)} \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\,. \qquad \text{(Ampére)} \tag{2.4}$$

The quantities appearing in these equations are as follows.

- $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ are the electric and magnetic fields, respectively.
- $\rho(t, \mathbf{x})$ is the electric charge density, and $\mathbf{J}(t, \mathbf{x})$ is the electric current density (defined so that $\mathbf{J} \cdot d\mathbf{S}$ is the charge per time passing through a stationary area element $d\mathbf{S}$).
- The constants $\epsilon_0 = 8.854187 \cdots \times 10^{-12}\,\mathrm{kg}^{-1}\,\mathrm{m}^{-3}\,\mathrm{s}^4\,\mathrm{A}^2$ (defined) is the electric permittivity of free space, and $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{kg\,m\,s}^{-2}\,\mathrm{A}^{-2}$ (definition) is the permeability of free space.

Via the force laws (see below), ϵ_0 and μ_0 serve to define the magnitudes of SI electrical units in terms of the units of mass, length and time. They are not independent since $\mu_0\epsilon_0$ has the units of $1/\text{speed}^2$ – indeed, as shown in IB *Electromagnetism*, $\mu_0\epsilon_0 = 1/c^2$. By convention, we choose μ_0 to take the value given here, which then defines the unit of current (ampere) and hence charge (coulomb or ampere second). Since we define the metre in terms of the second (the latter defined with reference to the frequency of a particular hyperfine transition in caesium) such that $c = 2.99792458 \times 10^8\,\mathrm{m\,s}^{-1}$, the value of ϵ_0 is then fixed as $1/(\mu_0 c^2)$.

The inclusion of the *displacement current* $\mu_0\epsilon_0 \partial\mathbf{E}/\partial t$ in Eq. (2.4), ensures that the continuity equation is satisfied, which expresses the conservation of electric charge:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0\,. \tag{2.5}$$

Maxwell's equations tell us how charges and currents generate electric and magnetic fields. We also need to know how charged particles move in electromagnetic fields, and this is governed by the *Lorentz force law*:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \,, \tag{2.6}$$

where q is the charge on the particle and $\mathbf{v}$ is its velocity. The continuum form of the force equation follows from summing over a set of charges; the force density (force per volume) is then

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \,, \tag{2.7}$$

in terms of the charge and current densities.

Maxwell's equations are a remarkable triumph of 19th century theoretical physics. In particular, they unify the phenomena of not only electricity and magnetism but also light. Despite pre-dating the development of *special relativity*, Maxwell's equations are fully consistent with this theory – the equations take the same form after Lorentz transformations if the electric and magnetic fields are transformed appropriately. As discussed in IB *Electromagnetism*, the relativistic covariance of Maxwell's equations is most easily seen by writing them in their 4D relativistic form:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \tag{2.8}$$

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0 \,. \tag{2.9}$$

The quantities appearing in these equations are as follows.

- The spacetime derivative $\partial_\mu \equiv \partial/\partial x^\mu$, where $x^\mu = (ct, \mathbf{x})$ are the spacetime coordinates in some inertial frame.
- The *Maxwell field-strength tensor* $F^{\mu\nu}$ is an antisymmetric type-$\binom{2}{0}$ tensor, whose components in some inertial frame are the electric and magnetic fields in that frame:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} . \tag{2.10}$$

  For reference, lowering the indices we have

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} . \tag{2.11}$$

- The *current 4-vector* has the 3D charge and current densities as its components:

$$J^\mu = (\rho c, \mathbf{J}) \,. \tag{2.12}$$

The current 4-vector satisfies the relativistic continuity equation

$$\partial_\mu J^\mu = 0\,, \tag{2.13}$$

which is consistent with Eq. (2.8) due to the antisymmetry of $F^{\mu\nu}$.

The first of the covariant Maxwell's equations (2.8) encodes the two sourced 3D equations (2.1) and (2.4). The other two source-free 3D equations are repackaged in Eq. (2.9). This equation is sometimes written in terms of the *dual field strength*

$$^\star F^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,, \tag{2.14}$$

where the alternating (pseudo-)tensor $\epsilon^{\mu\nu\rho\sigma}$ is the fully-antisymmetric object defined by

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0,1,2,3 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0,1,2,3 \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.15}$$

as

$$\partial_\mu {}^\star F^{\mu\nu} = 0\,. \tag{2.16}$$

(Note that $^\star F^{\mu\nu}$ is sometimes written as $\tilde{F}^{\mu\nu}$.)

The Maxwell equation (2.9) allows us to write the field strength as the antisymmetrised derivative of a 4-vector potential A^μ:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{2.17}$$

Writing the components as

$$A^\mu = (\phi/c, \mathbf{A})\,, \tag{2.18}$$

the *scalar potential* ϕ and the 3D *vector potential* $\mathbf{A}$ are related to the electric and magnetic fields by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi \tag{2.19}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}\,. \tag{2.20}$$

The $F^{\mu\nu}$, and hence the electric and magnetic fields, are unchanged if we add the gradient of a 4D scalar to A^μ, i.e.,

$$A^\mu \to A^\mu + \partial^\mu \chi\,. \tag{2.21}$$

Such transformations are known as *gauge transformations*. In 3D language, we have

$$\phi \to \phi - \partial\chi/\partial t \qquad \text{and} \qquad \mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi\,. \tag{2.22}$$

We shall make use of this gauge freedom later in Sec. 5 to simplify the calculation of the emission of electromagnetic radiation by time-dependent charge distributions.

Under a Lorentz transformation,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \,, \tag{2.23}$$

with

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta} \tag{2.24}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ is the *Minkowski metric*, the Maxwell field strength transforms as

$$F^{\mu\nu} \to \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta F^{\alpha\beta} \,. \tag{2.25}$$

The transformations of $\mathbf{E}$ and $\mathbf{B}$ can be read off from the components of this equation; for the special case of a standard *boost* along the x-axis with speed $v = \beta c$, for which

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.26}$$

with Lorentz factor $\gamma \equiv (1-\beta^2)^{-1/2}$, we have

$$\mathbf{E}' = \begin{pmatrix} E_x \\ \gamma(E_y - vB_z) \\ \gamma(E_z + vB_y) \end{pmatrix} \quad \text{and} \quad \mathbf{B}' = \begin{pmatrix} B_x \\ \gamma(B_y + vE_z/c^2) \\ \gamma(B_z - vE_y/c^2) \end{pmatrix} \,. \tag{2.27}$$

The electric and magnetic fields are generally mixed up by Lorentz transformations. There are, however, two *quadratic invariants* that are quadratic in the fields and do not change under Lorentz transformations. The first of these is simply

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 \,. \tag{2.28}$$

The second contracts $F_{\mu\nu}$ with its dual:

$$-\frac{1}{4}F_{\mu\nu}{}^\star F^{\mu\nu} = \frac{1}{c}\mathbf{E}\cdot\mathbf{B} \,. \tag{2.29}$$

The third contraction we might consider forming is ${}^\star F_{\mu\nu}{}^\star F^{\mu\nu}$. However, this does not generate a new invariant since ${}^\star F_{\mu\nu}{}^\star F^{\mu\nu} = F_{\mu\nu}F^{\mu\nu}$. Higher-order invariants such as $\det(F^\mu{}_\nu)$ are also sometimes useful.

Finally, we consider the Lorentz force law. Recall that in special relativity, the *momentum 4-vector* p^μ of a point particle is related to the *force 4-vector* F^μ by

$$dp^\mu/d\tau = F^\mu \,. \tag{2.30}$$

Here, τ is proper time for the particle and $p^\mu = (E/c, \mathbf{p})$, with E the energy of the particle and $\mathbf{p}$ its 3D momentum. For a particle of mass m, the momentum 4-vector is related to the *velocity 4-vector*

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{d}{dt}(ct, \mathbf{x}) = (\gamma c, \gamma \mathbf{v}) , \tag{2.31}$$

where $\mathbf{v}$ is the 3D velocity and γ is the associated Lorentz factor, by

$$p^\mu = m u^\mu . \tag{2.32}$$

The norm is constant, $p^\mu p_\mu = -m^2 c^2$, and the energy and momentum are related to the velocity by $E = \gamma m c^2$ and $\mathbf{p} = \gamma m \mathbf{v}$. If the particle has charge q, it experiences a force 4-vector due to the electromagnetic field given by $F^\mu = q F^{\mu\nu} u_\nu$, so that

$$dp^\mu / d\tau = q F^{\mu\nu} u_\nu . \tag{2.33}$$

Note that the antisymmetry of $F^{\mu\nu}$ ensures that the norm of p^μ is conserved. The spatial components of Eq. (2.33) are just the 3D Lorentz force law in Eq. (2.6), while the zero (time) component is

$$dE/dt = q \mathbf{E} \cdot \mathbf{v} , \tag{2.34}$$

which relates the rate of change of energy of the particle to the rate at work is being done on it by the electric field. The magnetic field does no work since it generates a force $q\mathbf{v} \times \mathbf{B}$ that is perpendicular to $\mathbf{v}$.

3 Action principles for electrodynamics

Recall that in non-relativistic classical mechanics, the equation of motion $m\ddot{\mathbf{x}} = -\boldsymbol{\nabla} V$ for a particle of mass m moving in a potential $V(t, \mathbf{x})$ may be derived by extremising the *action*

$$S = \int_{t_1}^{t_2} \left(\frac{1}{2} m \dot{\mathbf{x}}^2 - V(t, \mathbf{x}) \right) dt . \tag{3.1}$$

In other words, we look for the path $\mathbf{x}(t)$, with fixed end-points at times t_1 and t_2, for which the action is stationary under changes of path. Note that the action has dimensions of energy $\times$ time.

The action is the integral of the *Lagrangian*

$$L = \frac{1}{2} m \dot{\mathbf{x}}^2 - V(t, \mathbf{x}) . \tag{3.2}$$

The Euler–Lagrange equations,

$$\frac{\partial L}{\partial \mathbf{x}} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{x}}} \right) , \tag{3.3}$$

determine the stationary path as that satisfying

$$-\boldsymbol{\nabla} V = \frac{d}{dt}\left(m\dot{\mathbf{x}}\right) \tag{3.4}$$

(subject to the boundary conditions at t_1 and t_2). This is just the expected Newtonian equation of motion.

In this section, we shall generalise this action principle to relativistic motion of point particles moving in prescribed electromagnetic fields, and extend the principle to the field itself. The ultimate motivation for this is that the starting point for quantizing any classical theory is the action. The quantization of fields is not discussed in this course, but is fully developed in Part-III courses such as *Quantum Field Theory* and *Advanced Quantum Field Theory.*

3.1 Relativistic point particles in external fields

We start by seeking an action principle for a free relativistic point particle of rest mass m. We shall then include the effect of an external electromagnetic field.

The non-relativistic action for a free particle is just the integral of the kinetic energy $m\dot{\mathbf{x}}^2/2$ over time. A relativistic version of the action principle must involve a Lorentz-invariant action, and reduce to the non-relativistic version in the limit of speeds much less than c. A good candidate for the relativistic Lagrangian is $-mc^2/\gamma$, where γ is the Lorentz factor of the particle. To see this, note that $dt/\gamma = d\tau$ (the proper time increment) and so the action is Lorentz invariant – it is just the proper time along the path between the two fixed spacetime endpoints. Moreover,

$$\begin{aligned} -mc^2\left(1-\frac{\mathbf{v}^2}{c^2}\right)^{1/2} &= -mc^2\left(1-\frac{1}{2}\frac{\mathbf{v}^2}{c^2}+\cdots\right) \\ &= -mc^2+\frac{1}{2}m\mathbf{v}^2+\cdots\,, \end{aligned} \tag{3.5}$$

and so the action reduces to the usual Newtonian one in the non-relativistic limit up to an irrelevant constant ($-mc^2$). Writing the Lagrangian as

$$L = -mc^2\left(1-\mathbf{v}^2/c^2\right)^{1/2}\,, \tag{3.6}$$

the Euler–Lagrange equations give

$$\frac{d}{dt}\left(\frac{\mathbf{v}}{(1-\mathbf{v}^2/c^2)^{1/2}}\right) = 0\,, \tag{3.7}$$

since $\partial L/\partial \mathbf{x} = 0$. This is just the statement that the relativistic 3-momentum is conserved, as required for a free particle.

We now want to add in the effect of an external electromagnetic field. The non-relativistic form of the Lagrangian for a particle moving in a potential $V(\mathbf{x}, t)$ suggests adding $-q\phi$ to the relativistic Lagrangian, where q is the charge of the particle. However, this alone is insufficient since the magnetic field would not appear in the equation of motion. A further issue is related to gauge-invariance: the action principle must be invariant under gauge transformations of the form

$$\phi \to \phi - \frac{\partial \chi}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi\,. \tag{3.8}$$

This means that the *change* in action between two paths with fixed endpoints $\mathbf{x}_1$ and $\mathbf{x}_2$ at t_1 and t_2 must be gauge-invariant. We can fix both of these issues by including a further term $q\mathbf{A}\cdot\mathbf{v}$ in the Lagrangian:

$$L = -mc^2\left(1 - \mathbf{v}^2/c^2\right)^{1/2} - q\phi + q\mathbf{A}\cdot\mathbf{v}\,. \tag{3.9}$$

To see how this gives a gauge-invariant action principle, note that under a gauge transformation

$$\begin{aligned} S &\to S + q\int_{t_1}^{t_2}\left(\frac{\partial\chi}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\chi\right)dt \\ &= S + q\int_{t_1}^{t_2}\frac{d\chi}{dt}\,dt \\ &= S + q\left[\chi(t_2, \mathbf{x}_2) - \chi(t_1, \mathbf{x}_1)\right]\,. \end{aligned} \tag{3.10}$$

As the endpoints are fixed, the gauge transformation does not affect the change in the action (although it does change the absolute value of the action).

Finally, we need to check that we recover the correct equation of motion. The ith component of the Euler–Lagrange equations gives

$$\frac{d}{dt}\left(\frac{mv_i}{(1-\mathbf{v}^2/c^2)^{1/2}}\right) + q\frac{dA_i}{dt} = -q\frac{\partial\phi}{\partial x_i} + qv_j\frac{\partial A_j}{\partial x_i}\,. \tag{3.11}$$

Expanding the total (convective) derivative dA_i/dt gives the required equation of motion:

$$\begin{aligned} \frac{dp_i}{dt} &= -q\frac{\partial\phi}{\partial x_i} + qv_j\frac{\partial A_j}{\partial x_i} - q\frac{\partial A_i}{\partial t} - qv_j\frac{\partial A_i}{\partial x_j} \\ &= qE_i + q\left[\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})\right]_i \\ &= q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_i\,, \end{aligned} \tag{3.12}$$

where $\mathbf{p} = \gamma m\mathbf{v}$.

We can formulate the action principle more covariantly by recalling that the kinetic part of the action is just proportional to the proper time along the path, and can be written as

$$-mc^2 \int d\tau = -mc \int (-\eta_{\mu\nu} dx^\mu dx^\nu)^{1/2}$$
$$= -mc \int \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda\,, \tag{3.13}$$

for some arbitrary parameter λ along the path. This action is invariant to a reparameterisation of the path with a different λ[1]. We must now look for spacetime paths that extremise the action between fixed endpoint events. For the electromagnetic part of the action, note that

$$A_\mu dx^\mu = (-\phi + \mathbf{v} \cdot \mathbf{A})\, dt\,, \tag{3.14}$$

so the full action can be written covariantly as

$$S = -mc \int \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda + q \int A_\mu \frac{dx^\mu}{d\lambda}\, d\lambda\,. \tag{3.15}$$

It is straightforward to verify the gauge-invariance of changes in the action now since, under $A_\mu \to A_\mu + \partial_\mu \chi$, we have

$$A_\mu \frac{dx^\mu}{d\lambda} \to A_\mu \frac{dx^\mu}{d\lambda} + \frac{d\chi}{d\lambda}\,. \tag{3.16}$$

The last term on the right is a total derivative and so does not contribute to changes in the action for fixed endpoints.

For the covariant action, the paths are parameterised by λ so the relevant Euler–Lagrange equations are

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial (dx^\mu/d\lambda)}\right)\,, \tag{3.17}$$

where the Lagrangian

$$L = -mc \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\right)^{1/2} + qA_\mu \frac{dx^\mu}{d\lambda}\,. \tag{3.18}$$

The required derivatives of the Lagrangian are

$$\frac{\partial L}{\partial x^\mu} = q(\partial_\mu A_\nu)\frac{dx^\nu}{d\lambda}\,, \tag{3.19}$$

and

$$\frac{\partial L}{\partial (dx^\mu/d\lambda)} = \frac{mc}{[-\eta_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)]^{1/2}} \eta_{\mu\nu} \frac{dx^\nu}{d\lambda} + qA_\mu\,. \tag{3.20}$$

[1] It is more convenient to use a general parameter than τ itself since the latter would imply the *constraint* $\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) = -c^2$, which would have to be accounted for when varying the path.

Substituting these derivatives into the Euler–Lagrange equations, and using $dA_\mu/d\lambda = (dx^\nu/d\lambda)\partial_\nu A_\mu$, we find

$$
\begin{aligned}
mc\frac{d}{d\lambda}\left(\frac{\eta_{\mu\nu}dx^\nu/d\lambda}{\left[-\eta_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)\right]^{1/2}}\right) &= q\frac{dx^\nu}{d\lambda}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\
\Rightarrow m\eta_{\mu\nu}\frac{d}{d\lambda}\left(\frac{dx^\nu}{d\tau}\right) &= qF_{\mu\nu}\frac{dx^\nu}{d\lambda}\,.
\end{aligned} \tag{3.21}
$$

This equation is reparameterisation invariant, as expected since the action is. Writing $d/d\lambda = (d\tau/d\lambda)d/d\tau$, and noting that λ is arbitrary, we recover

$$
\frac{dp_\mu}{d\tau} = qF_{\mu\nu}u^\nu\,, \tag{3.22}
$$

where p^μ is the 4-momentum and $u^\mu = dx^\mu/d\tau$ is the 4-velocity. This is the expected covariant equation of motion (Eq. 2.33).

3.1.1 Motion of charged particles in constant, uniform fields

We now make a brief interlude to discuss the motion of a point particle of rest mass m and charge q in uniform electric and magnetic fields. The field-strength tensor is independent of spacetime position and the equation of motion (2.33) becomes

$$
\frac{du^\mu}{d\tau} = \frac{q}{m}F^\mu{}_\nu u^\nu\,, \tag{3.23}
$$

with $u^\mu = dx^\mu/d\tau$. The mixed components of the field-strength tensor are

$$
F^\mu{}_\nu = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}. \tag{3.24}
$$

The solution of Eq. (3.23) with initial condition $u^\mu = u^\mu(0)$ at $\tau = 0$ is

$$
u^\mu(\tau) = \exp\left(\frac{q\tau}{m}F\right)^\mu{}_\nu u^\nu(0)\,, \tag{3.25}
$$

where, as usual, the exponential is defined by its power series:

$$
\exp\left(\frac{q\tau}{m}F\right)^\mu{}_\nu = \delta^\mu_\nu + \frac{q\tau}{m}F^\mu{}_\nu + \frac{1}{2}\left(\frac{q\tau}{m}\right)^2 F^\mu{}_\rho F^\rho{}_\nu + \cdots\,. \tag{3.26}
$$

Integrating again gives the worldline

$$
x^\mu(\tau) = x^\mu(0) + \left[\int_0^\tau \exp\left(\frac{q\tau'}{m}F\right)^\mu{}_\nu\, d\tau'\right] u^\nu(0)\,, \tag{3.27}
$$

where the spacetime position at $\tau = 0$ is $x^\mu(0)$. Now, for an invertible matrix $\mathbf{M}$, we have

$$\begin{aligned}\int_0^\tau \exp(\mathbf{M}\tau')\, d\tau' &= \int_0^\tau \left(\mathbf{I} + \mathbf{M}\tau' + \frac{1}{2}\mathbf{M}^2\tau'^2 + \frac{1}{3!}\mathbf{M}^3\tau'^3 + \cdots\right) d\tau' \\ &= \tau\mathbf{I} + \frac{1}{2}\mathbf{M}\tau^2 + \frac{1}{3!}\mathbf{M}^2\tau^3 + \frac{1}{4!}\mathbf{M}^3\tau^4 + \cdots \\ &= \mathbf{M}^{-1}\left(\mathbf{I} + \mathbf{M}\tau + \frac{1}{2}\mathbf{M}^2\tau^2 + \frac{1}{3!}\mathbf{M}^3\tau^3 + \cdots - \mathbf{I}\right) \\ &= \mathbf{M}^{-1}\left[\exp(\mathbf{M}\tau) - \mathbf{I}\right] . \end{aligned} \tag{3.28}$$

With a suitable choice of spacetime origin, we can therefore write

$$x^\mu(\tau) = \frac{m}{q}(F^{-1})^\mu{}_\nu \exp\left(\frac{q\tau}{m}F\right)^\nu{}_\rho u^\rho(0) \quad \text{for } \det(F^\mu{}_\nu) \neq 0 . \tag{3.29}$$

Exercise: Show that the quartic invariant

$$\det(F^\mu{}_\nu) = -(\mathbf{E}\cdot\mathbf{B})^2/c^2 . \tag{3.30}$$

We first consider the case where $\mathbf{E}\cdot\mathbf{B} \neq 0$, i.e., both $\mathbf{E}$ and $\mathbf{B}$ are non-zero and are not perpendicular. (Note that this holds in all inertial frames, if true in one, due to the Lorentz invariance of $\mathbf{E}\cdot\mathbf{B}$.) We shall now show that a frame can always be found in which the $\mathbf{E}$ and $\mathbf{B}$ fields are parallel. The motion turns out to be rather simple in this frame, and the motion in a general frame can then by found by an appropriate Lorentz transformation.

For the case that $\mathbf{E}$ and $\mathbf{B}$ are not parallel in S, let us choose spatial axes such that both fields lie in the y-z plane. It follows that under a boost along the x-axis, the transformed fields are still in the y'-z' plane with components

$$\mathbf{E}' = \gamma \begin{pmatrix} 0 \\ E_y - vB_z \\ E_z + vB_y \end{pmatrix} \quad \text{and} \quad \mathbf{B}' = \gamma \begin{pmatrix} 0 \\ B_y + vE_z/c^2 \\ B_z - vE_y/c^2 \end{pmatrix} . \tag{3.31}$$

Since we want to demonstrate that for some $\beta = v/c$ with $|\beta| < 1$, we can set $\mathbf{E}'$ and $\mathbf{B}'$ parallel, consider $\mathbf{E}'\times\mathbf{B}'$ (which lies along the x'-axis):

$$\begin{aligned}(\mathbf{E}'\times\mathbf{B}')_x &= \gamma^2\left[(E_y - vB_z)(B_z - vE_y/c^2) - (E_z + vB_y)(B_y + vE_z/c^2)\right] \\ &= \gamma^2\left[(\mathbf{E}\times\mathbf{B})_x(1+\beta^2) - v|\mathbf{B}|^2 - v|\mathbf{E}/c|^2\right] . \end{aligned} \tag{3.32}$$

It follows that if we choose the velocity such that

$$\frac{\beta}{1+\beta^2} = \frac{(\mathbf{E}\times\mathbf{B})_x/c}{|\mathbf{B}|^2+|\mathbf{E}/c|^2}\,, \tag{3.33}$$

then the electric and magnetic fields will be parallel in S'. However, this is only possible if we can solve Eq. (3.33) for $|\beta|<1$. For given $|\mathbf{E}|$ and $|\mathbf{B}|$, the magnitude $|(\mathbf{E}\times\mathbf{B})_x| < |\mathbf{E}||\mathbf{B}|$ since $\mathbf{E}$ and $\mathbf{B}$ are not perpendicular. Writing $|\mathbf{B}| = \mu|\mathbf{E}|/c$ for some dimensionless $\mu > 0$, it follows that the magnitude of the right-hand side of Eq. (3.33) satisfies

$$\left|\frac{(\mathbf{E}\times\mathbf{B})_x/c}{|\mathbf{B}|^2+|\mathbf{E}/c|^2}\right| < \frac{\mu}{1+\mu^2}\,. \tag{3.34}$$

The maximum value of the function $\mu/(1+\mu^2)$ is $1/2$ (at $\mu = 1$) and so the right-hand side of Eq. (3.33) is necessarily less than $1/2$. It follows that we can always find a β with $|\beta|<1$, such that Eq. (3.33) is satisfied and the transformed electric and magnetic fields are parallel.

Adopting a frame in which $\mathbf{E}$ and $\mathbf{B}$ are parallel, and now taking these to lie along the x-axis, we have $\mathbf{E} = (E,0,0)$ and $\mathbf{B} = (B,0,0)$ so that the field-strength tensor takes the block-diagonal form

$$F^\mu{}_\nu = \begin{pmatrix} 0 & E/c & 0 & 0 \\ E/c & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}\,. \tag{3.35}$$

Noting that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \mathbf{I} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -\mathbf{I}\,, \tag{3.36}$$

we have

$$\begin{aligned} \exp\begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix} &= \mathbf{I} + w\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{w^2}{2}\mathbf{I} + \frac{w^3}{3!}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{w^4}{4!}\mathbf{I} + \cdots \\ &= \cosh w\mathbf{I} + \sinh w\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} \exp\begin{pmatrix} 0 & w' \\ -w' & 0 \end{pmatrix} &= \mathbf{I} + w'\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{w'^2}{2}\mathbf{I} - \frac{w'^3}{3!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{w'^4}{4!}\mathbf{I} + \cdots \\ &= \cos w'\mathbf{I} + \sin w'\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,. \end{aligned} \tag{3.38}$$

It follows that

$$\exp\left(\frac{q\tau}{m}F\right)^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\left(\frac{q\tau}{mc}E\right) & \sinh\left(\frac{q\tau}{mc}E\right) & 0 & 0 \\ \sinh\left(\frac{q\tau}{mc}E\right) & \cosh\left(\frac{q\tau}{mc}E\right) & 0 & 0 \\ 0 & 0 & \cos\left(\frac{q\tau}{m}B\right) & \sin\left(\frac{q\tau}{m}B\right) \\ 0 & 0 & -\sin\left(\frac{q\tau}{m}B\right) & \cos\left(\frac{q\tau}{m}B\right) \end{pmatrix}, \qquad (3.39)$$

which, from Eq. (3.29), gives the motion

$$\begin{aligned} ct(\tau) &= \frac{mc}{qE}\left[\sinh\left(\frac{q\tau}{mc}E\right)c\dot{t}(0) + \cosh\left(\frac{q\tau}{mc}E\right)\dot{x}(0)\right] \\ x(\tau) &= \frac{mc}{qE}\left[\cosh\left(\frac{q\tau}{mc}E\right)c\dot{t}(0) + \sinh\left(\frac{q\tau}{mc}E\right)\dot{x}(0)\right] \\ y(\tau) &= \frac{m}{qB}\left[\sin\left(\frac{q\tau}{m}B\right)\dot{y}(0) - \cos\left(\frac{q\tau}{m}B\right)\dot{z}(0)\right] \\ z(\tau) &= \frac{m}{qB}\left[\cos\left(\frac{q\tau}{m}B\right)\dot{y}(0) + \sin\left(\frac{q\tau}{m}B\right)\dot{z}(0)\right]. \end{aligned} \qquad (3.40)$$

Notice how, with **E** and **B** parallel, the motion in the ct-x plane decouples from that in the y-z plane. The ct-x motion is a hyperbolic trajectory (like for the example of constant rest-frame acceleration given in Appendix A) with

$$x^2 - (ct)^2 = \left(\frac{mc}{qE}\right)^2 \left([c\dot{t}(0)]^2 - \dot{x}^2(0)\right) > 0\,, \qquad (3.41)$$

where the inequality follows from $[c\dot{t}(0)]^2 - \dot{\mathbf{x}}^2(0) = c^2$. Note that

$$[c\dot{t}(\tau)]^2 - \dot{x}^2(\tau) = [c\dot{t}(0)]^2 - \dot{x}^2(0)\,. \qquad (3.42)$$

The motion in the y-z plane is circular with

$$y^2 + z^2 = \left(\frac{m}{qB}\right)^2 \left(\dot{y}^2(0) + \dot{z}^2(0)\right)\,. \qquad (3.43)$$

The radius of the circle is controlled by the transverse proper-time speed $(\dot{y}^2 + \dot{z}^2)^{1/2}$ which is constant since

$$\dot{y}^2(\tau) + \dot{z}^2(\tau) = \dot{y}^2(0) + \dot{z}^2(0)\,. \qquad (3.44)$$

Defining the *cyclotron frequency* $\omega_c \equiv qB/m$, we see that this is the angular frequency of the circular motion with respect to proper time. Combining the two motions gives, generally, a helical motion along the field direction. The radius of the circular projection is constant in time, but the spacing along the x-axis grows in time and asymptotically the spacing is constant in $\ln x$. As the speed along the x-axis approaches c, the period of the circular motion in t gets significantly dilated and the transverse speed $[(dy/dt)^2 + (dz/dt)^2]^{1/2}$ falls towards zero.

To complete our treatment of motion in constant, uniform fields we now consider the special cases for which $\mathbf{E}\cdot\mathbf{B}=0$.

$\mathbf{B}=0$. The motion can be obtained as the limit of Eq. (3.40) as $B\to 0$ (with an appropriate shift in the choice of origin which is infinite in the limit). The worldline is hyperbolic in the ct-x plane, but now linear in the y-z plane corresponding to circular motion in that plane with infinite radius. The velocity in the y-z plane (perpendicular to the electric field) asymptotes to zero.

$\mathbf{E}=0$. The motion can be obtained as the limit of Eq. (3.40) as $E\to 0$. The motion in the ct-x plane is linear with $c\dot{t}$ and $\dot{x}$ constant, corresponding to a constant 3-velocity component along the x-axis (the direction of the magnetic field). Constancy of $c\dot{t}$ ensures that the speed of the particle is constant, as required since the magnetic field does no work. In the y-z plane, the motion is circular with proper angular frequency ω_c. If the 3-velocity along the field direction is v_x, and the perpendicular velocity is $\mathbf{v}_\perp$, the constant Lorentz factor is $\gamma^{-2}=1-v_x^2/c^2-\mathbf{v}_\perp^2/c^2$ (and $\dot{t}=\gamma$) and the angular frequency of the circular motion with respect to coordinate time t is ω_c/γ. The spacing of the helical motion along the x-axis is constant. The radius r of the circular motion follows from

$$r^2=\frac{1}{\omega_c^2}\left(\dot{y}^2(0)+\dot{z}^2(0)\right)=\frac{\gamma^2\mathbf{v}_\perp^2}{\omega_c^2}\,. \tag{3.45}$$

This simple helical motion can be easily obtained directly from the equations of motion in 3D form. Since the speed is constant, $d\mathbf{p}/dt=\gamma m d\mathbf{v}/dt$ and the Lorentz force law gives

$$\frac{d\mathbf{v}}{dt}=\frac{q}{\gamma m}\mathbf{v}\times\mathbf{B}\,, \tag{3.46}$$

with $\gamma^{-2}=1-\mathbf{v}^2/c^2$. It follows that the component of $\mathbf{v}$ along the field is constant, while the perpendicular component precesses on a circle with angular frequency $\omega=qB/(\gamma m)$. The radius of the circle then follows from $|\mathbf{v}_\perp|=\omega r$.

$\mathbf{E}\perp\mathbf{B}$ *with non-zero* $\mathbf{E}$ *and* $\mathbf{B}$. We can always choose axes such that $\mathbf{E}=(0,E,0)$ and $\mathbf{B}=(0,0,B)$ (where E and B may be negative). If we now boost along the x-axis, we have

$$\mathbf{E}'=\begin{pmatrix}0\\ \gamma(E-vB)\\ 0\end{pmatrix}\quad\text{and}\quad\mathbf{B}'=\begin{pmatrix}0\\ 0\\ \gamma(B-vE/c^2)\end{pmatrix}. \tag{3.47}$$

If $|\mathbf{E}|>c|\mathbf{B}|$, we can set $\mathbf{B}'=0$ by taking $\beta=cB/E$. We are then left with motion in a uniform electric field, which was discussed above. For $|\mathbf{E}|<c|\mathbf{B}|$, we can set $\mathbf{E}'=0$ by taking $\beta=E/(cB)$ and we have helical motion in a constant magnetic field. Since a particle at rest in a magnetic field remains stationary, there exists a solution for $|\mathbf{E}|<c|\mathbf{B}|$ where in the original frame the particle moves at constant velocity in the direction perpendicular to the crossed fields. This is simply the velocity $\mathbf{v}$ such that electric and magnetic forces cancel, i.e., $q\mathbf{E}=-q\mathbf{v}\times\mathbf{B}$ requiring $\mathbf{v}=\mathbf{E}\times\mathbf{B}/\mathbf{B}^2$. The

fact that charged particles with a specific velocity can move uniformly through crossed electric and magnetic fields can be used to select charged particles on their velocity. The special case $|\mathbf{E}| = c|\mathbf{B}|$ is left as an exercise.

Exercise: For perpendicular magnetic fields with $|\mathbf{E}| = c|\mathbf{B}|$ and $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, E/c)$, show that $ct(\tau)$ and $x(\tau)$ are, generally, cubic polynomials in τ, $y(\tau)$ is a quadratic polynomial, and $z(\tau)$ is linear.

3.2 Action principle for the electromagnetic field

We now extend the action principle to the electromagnetic field itself. Rather than dealing with functions of a single variable, like $x^\mu(\lambda)$, we now have fields defined over spacetime. The action will therefore involve the integral over spacetime (with the Lorentz-invariant measure[2] $d^4x = cdt\, d^3\mathbf{x}$) of a Lorentz-invariant *Lagrangian density* $\mathcal{L}$. This Lagrangian density must also be gauge-invariant. The variational principle is then to find field configurations such that the action is stationary for arbitrary changes in the field that vanish on some closed surface in spacetime.

3.2.1 Action principle with a prescribed current

We first develop the action principle for the electromagnetic field in the presence of a prescribed 4-vector current density $J^\mu(x)$ over spacetime. Of course, the current must be conserved: $\partial_\mu J^\mu = 0$. We shall then see how to combine this action with that for point particles to obtain the full action of classical electrodynamics.

The covariant Maxwell equations involve first derivatives of $F_{\mu\nu}$ or, equivalently, second derivatives of A_μ. Moreover, we need a Lagrangian density that is a Lorentz-invariant scalar. Recalling the quadratic invariant (2.28), we consider a Lagrangian density $\mathcal{L} \propto F_{\mu\nu}F^{\mu\nu}$, with A_μ the underlying field to vary, for the part of the action describing the free electromagnetic field. Note that, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the Maxwell equation (2.9) becomes an identity.

We also need to include an interaction term between the field and the current to generate the source term in the Maxwell equation (2.8). A suitable contribution to the

[2]Recall that the Lorentz transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ has $(\det \Lambda^\mu{}_\nu)^2 = 1$ so the Jacobian of the transformation relating d^4x and d^4x' is unity.

Lagrangian density is $\mathcal{L} \propto A_\mu J^\mu$. We therefore consider an action of the form

$$S = -\frac{1}{4\mu_0 c}\int F_{\mu\nu}F^{\mu\nu}\,d^4x + \frac{1}{c}\int A_\mu J^\mu\,d^4x\,. \tag{3.48}$$

Note that the first term in the action, describing the free electromagnetic field, has the correct dimensions of energy × time. The numerical factor is chosen to get the correct field equation (see below). The interaction term is not gauge-invariant since, under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$,

$$\begin{aligned}\int A_\mu J^\mu\,d^4x \to& \int A_\mu J^\mu\,d^4x + \int \partial_\mu \chi J^\mu\,d^4x \\ =& \int A_\mu J^\mu\,d^4x + \int \partial_\mu\left(\chi J^\mu\right)\,d^4x\,,\end{aligned} \tag{3.49}$$

where we have used current conservation. The last term is, however, a total derivative and so only contributes a boundary term to the action; it does not contribute to the variation of the action between neighbouring field configurations.

Varying the action (3.48) with respect to A_μ we have

$$\begin{aligned}\delta S &= -\frac{2}{4\mu_0 c}\int(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)F^{\mu\nu}\,d^4x + \frac{1}{c}\int \delta A_\mu J^\mu\,d^4x \\ &= -\frac{1}{\mu_0 c}\int\left(F^{\mu\nu}\partial_\mu \delta A_\nu - \mu_0 J^\nu \delta A_\nu\right)\,d^4x \\ &= \frac{1}{\mu_0 c}\int\left(\partial_\mu F^{\mu\nu} + \mu_0 J^\nu\right)\delta A_\nu\,d^4x\,.\end{aligned} \tag{3.50}$$

Here, we have integrated by parts going to the last equality and dropped the surface term since the variation δA_ν vanishes on the boundary. If the action is at an extremum, the change δS must vanish *for all* δA_μ. This can only be the case if

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu\,, \tag{3.51}$$

which is the desired (sourced) Maxwell equation.

3.2.2 Full action for classical electrodynamics

We now combine the action for a point particle moving in an external field with that given above for the electromagnetic field itself. We have to be careful not to include the interaction terms twice – indeed, the interaction term in Eq. (3.48) is exactly the same as that already included in Eq. (3.15). To see this, note that the current 4-vector for a point particle of charge q following a wordline $y^\mu(\lambda)$ can be written covariantly as (see below)

$$J^\mu(x) = qc\int \delta^{(4)}\left(x - y(\lambda)\right)\frac{dy^\mu}{d\lambda}\,d\lambda\,. \tag{3.52}$$

The 4D delta-function here is Lorentz-invariant (since the 4D measure d^4x is), and its appearance ensures that the integration over λ only selects that event on the wordline $y(\lambda)$ (if any) for which $y^\mu(\lambda) = x^\mu$. It follows that

$$\int A_\mu J^\mu \, d^4x = qc \int A_\mu \left(y(\lambda)\right) \frac{dy^\mu}{d\lambda} \, d\lambda \,, \tag{3.53}$$

which establishes the equivalence of the interaction terms.

Current 4-vector of a point particle: We can establish Eq. (3.52) by writing the 4D delta-function as

$$\delta^{(4)}\left(x - y(\lambda)\right) = \delta\left(ct - y^0(\lambda)\right) \delta^{(3)}\left(\mathbf{x} - \mathbf{y}(\lambda)\right) \,, \tag{3.54}$$

and using the following general result for the 1D delta function with a function $f(x)$ as its argument:

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x - x_{*,i})}{|f'(x_{*,i})|} \,. \tag{3.55}$$

Here, the sum is over the roots $x_{*,i}$ of $f(x) = 0$, i.e., $f(x_{*,i}) = 0$, and $f'(x)$ is the derivative of $f(x)$. Integrating out the first delta-function in Eq. (3.54), considered as a function of λ, with Eq. (3.55), we have

$$J^\mu(x) = qc\delta^{(3)}\left(\mathbf{x} - \mathbf{y}(t)\right) \dot{y}^\mu / \dot{y}^0 \,, \tag{3.56}$$

where $\mathbf{y}(t)$ is the 3D location of the charge at coordinate time t. In components,

$$\rho(t, \mathbf{x}) = q\delta^{(3)}\left(\mathbf{x} - \mathbf{y}(t)\right) \tag{3.57}$$

$$\mathbf{J}(t, \mathbf{x}) = q\mathbf{v}(t)\delta^{(3)}\left(\mathbf{x} - \mathbf{y}(t)\right) \,, \tag{3.58}$$

where $\mathbf{v}(t)$ is the 3D velocity of the charge at time t. These are clearly the correct (non-covariant) expressions for the charge density and current density due to a point charge.

We are therefore led to consider a total action of the form

$$S = -\frac{1}{4\mu_0 c} \int F_{\mu\nu} F^{\mu\nu} \, d^4x - mc \int \left(-\eta_{\mu\nu} \frac{dy^\mu}{d\lambda} \frac{dy^\nu}{d\lambda}\right)^{1/2} d\lambda + q \int A_\mu \left(y(\lambda)\right) \frac{dy^\mu}{d\lambda} \, d\lambda \,, \tag{3.59}$$

which we vary with respect to $A_\mu(x)$ and the wordline $y^\mu(\lambda)$. Varying $y^\mu(\lambda)$ gives the equation of motion (3.22), while varying with respect to A_μ gives the Maxwell equation (2.8) with the current 4-vector given by Eq. (3.52). The action in Eq. (3.59) can easily be extended to a system of point particles by summing over particles in the second and third terms.

4 Energy and momentum of the electromagnetic field

In this section we shall discuss the energy and momentum densities carried by the electromagnetic field, and show how these are combined with the fluxes of energy and momentum into the spacetime *stress-energy tensor*.

4.1 Energy and momentum conservation

We begin by recalling Poynting's theorem, which expresses energy conservation in electromagnetism. In any inertial frame, the rate of work done per unit volume by the electromagnetic field on charged particles is $\mathbf{J}\cdot\mathbf{E}$. Note that magnetic field does no work since the magnetic force on any charge is perpendicular to its velocity. We now eliminate $\mathbf{J}$ in terms of the fields using Eq. (2.4) to find

$$\mathbf{J}\cdot\mathbf{E} = \frac{1}{\mu_0}\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\,. \tag{4.1}$$

We can manipulate this further using

$$\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) - \mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,. \tag{4.2}$$

Replacing $\boldsymbol{\nabla}\times\mathbf{E}$ with $-\partial\mathbf{B}/\partial t$ from Eq. (2.2), and substituting in Eq. (4.1), we find

$$\mathbf{J}\cdot\mathbf{E} = -\frac{\partial}{\partial t}\left(\frac{\epsilon_0\mathbf{E}^2}{2} + \frac{\mathbf{B}^2}{2\mu_0}\right) - \frac{1}{\mu_0}\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})\,. \tag{4.3}$$

Introducing the *Poynting vector*,

$$\mathbf{N} \equiv \mathbf{E}\times\mathbf{B}/\mu_0\,, \tag{4.4}$$

and recalling the electromagnetic energy density (derived for electrostatics and magnetostatics in IB *Electromagnetism*)

$$\epsilon = \frac{\epsilon_0\mathbf{E}^2}{2} + \frac{\mathbf{B}^2}{2\mu_0}\,, \tag{4.5}$$

we have the (non-)conservation law

$$\frac{\partial\epsilon}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{N} = -\mathbf{J}\cdot\mathbf{E}\,. \tag{4.6}$$

The Poynting vector therefore gives the *energy flux* of the electromagnetic field, i.e., the rate at which electromagnetic energy flows through a surface element $d\mathbf{S}$ is $\mathbf{N}\cdot d\mathbf{S}$. Integrating Eq. (4.6) over a volume V gives

$$\frac{d}{dt}\int_V \epsilon\, d^3\mathbf{x} + \int_V \mathbf{E}\cdot\mathbf{J}\, d^3\mathbf{x} = -\int_{\partial V}\mathbf{N}\cdot d\mathbf{S}\,. \tag{4.7}$$

In words, this states that the rate of change of electromagnetic energy in the volume plus the rate at which the field does work on charges in the volume (i.e., the rate of change of mechanical energy in the volume) is equal to the rate at which electromagnetic energy flows *into* the volume.

We now perform a similar calculation for the force per unit volume on charged particles, $\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}$. Eliminating ρ with $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0$ and $\mathbf{J}$ with Eq. (2.2) as above, we have the force density

$$\begin{aligned}\rho\mathbf{E} + \mathbf{J}\times\mathbf{B} &= \epsilon_0\mathbf{E}\boldsymbol{\nabla}\cdot\mathbf{E} + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} \\ &= \epsilon_0\mathbf{E}\boldsymbol{\nabla}\cdot\mathbf{E} + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \epsilon_0\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t} \\ &= \epsilon_0\mathbf{E}\boldsymbol{\nabla}\cdot\mathbf{E} + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \epsilon_0(\boldsymbol{\nabla}\times\mathbf{E})\times\mathbf{E}\,. \end{aligned} \tag{4.8}$$

We can make the right-hand side look more symmetric between $\mathbf{E}$ and $\mathbf{B}$ by adding in $\mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{B}/\mu_0 = 0$. Moreover, we can manipulate the curl terms with

$$(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} = \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B} - \boldsymbol{\nabla}(\mathbf{B}^2)/2\,, \tag{4.9}$$

and similarly for $\mathbf{B}\to\mathbf{E}$, to give

$$\begin{aligned}\rho\mathbf{E} + \mathbf{J}\times\mathbf{B} = \epsilon_0&\left(\mathbf{E}\boldsymbol{\nabla}\cdot\mathbf{E} + \mathbf{E}\cdot\boldsymbol{\nabla}\mathbf{E} - \frac{1}{2}\boldsymbol{\nabla}\mathbf{E}^2\right) + \frac{1}{\mu_0}\left(\mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{B} + \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B} - \frac{1}{2}\boldsymbol{\nabla}\mathbf{B}^2\right) \\ &- \epsilon_0\frac{\partial(\mathbf{E}\times\mathbf{B})}{\partial t}\,. \end{aligned} \tag{4.10}$$

The first two terms on the right-hand side are total derivatives; they are (minus) the divergence of the symmetric *Maxwell stress tensor*

$$\sigma_{ij} \equiv -\epsilon_0\left(E_iE_j - \frac{1}{2}\mathbf{E}^2\delta_{ij}\right) - \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\mathbf{B}^2\delta_{ij}\right)\,. \tag{4.11}$$

Further introducing the electromagnetic *momentum density*

$$\mathbf{g} \equiv \epsilon_0\mathbf{E}\times\mathbf{B} = \mathbf{N}/c^2\,, \tag{4.12}$$

we have a (non-)conservation law for momentum:

$$\frac{\partial g_i}{\partial t} + \frac{\partial\sigma_{ij}}{\partial x_j} = -\left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right)_i\,. \tag{4.13}$$

The Maxwell stress tensor σ_{ij} encodes the flux of 3-momentum. In particular, the rate at which the ith component of 3-momentum flows through a surface element $d\mathbf{S}$ is given by $\sigma_{ij}dS_j$. Equation (4.13) can be written in integral form as

$$\frac{d}{dt}\int_V g_i\,d^3\mathbf{x} + \int_V(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})_i\,d^3\mathbf{x} = -\int_{\partial V}\sigma_{ij}\,dS_j\,, \tag{4.14}$$

which states that the rate of change of electromagnetic momentum in some volume V, plus the total force on charged particles (i.e., the rate of change of mechanical momentum) is equal to the rate at which electromagnetic momentum flows into V through the boundary.

Note that the Maxwell stress tensor is symmetric. This is true more generally (e.g., it holds for the stress tensor in continuum and fluid mechanics) as a consequence of angular momentum conservation; we shall briefly review this argument in Sec. 4.5. Note also that the momentum density and energy flux (Poynting vector) are related, $\mathbf{g} = \mathbf{N}/c^2$. We might have anticipated this result by thinking quantum mechanically of the electromagnetic field being composed of massless photons (all moving at the speed of light), with energy and momentum related by $E = pc$. A net energy flux will then necessarily be accompanied by a momentum density with $\mathbf{g} = \mathbf{N}/c^2$. In fact, the relation between momentum density and energy flux is universal and is necessary to ensure symmetry of σ_{ij} in every inertial frame (see also Sec. 4.5).

Given the integral form of the conservation law, Eq. (4.14), we note finally that for static situations, the total electromagnetic force on a charge and current distribution can be calculated by integrating the Maxwell stress tensor over any surface that fully encloses the sources.

4.2 Stress-energy tensor

We now develop a covariant formulation of the energy and momentum conservation laws derived above, seeking to unite them in a single spacetime conservation law. In doing so, we shall show that ϵ, $\mathbf{N}$ (and $\mathbf{g}$) and σ_{ij} are the components of a type-$\binom{2}{0}$ 4D tensor, the *stress-energy tensor*.

To motivate the introduction of the stress-energy tensor, first consider a simpler mechanical example. Massive particles, of rest mass m, are all at rest in some inertial frame S with (proper) number density n_0. In that frame, we clearly have

$$\epsilon = mc^2 n_0 \,, \quad \mathbf{N} = 0 = \mathbf{g} \,, \quad \sigma_{ij} = 0 \,. \tag{4.15}$$

Now consider performing a standard Lorentz boost to the frame S' in which the particles have 3-velocity $\mathbf{v}' = -(v, 0, 0)$. In S', the number density is γn_0 (by length contraction), the energy of each particle is γmc^2 and the momentum of each is $\gamma m\mathbf{v}'$. It follows that the energy density in S' is

$$\epsilon' = \gamma mc^2 \gamma n_0 = \gamma^2 \epsilon \,. \tag{4.16}$$

There is now a non-zero energy flux in S', with

$$N'_x = -\epsilon' v = -\gamma^2 v \epsilon \,, \tag{4.17}$$

and a momentum density with non-zero component

$$g'_x = (-\gamma m v)\gamma n_0 = -\gamma^2 v\epsilon/c^2 = N'_x/c^2\,. \tag{4.18}$$

Finally, there is a flux of the x-component of 3-momentum along the x'-direction, so the stress tensor has a single non-zero component

$$\sigma'_{xx} = -v g'_x = \gamma^2 (v/c)^2\epsilon\,. \tag{4.19}$$

Note how ϵ, $\mathbf{N}$ and σ_{ij} mix amongst themselves under Lorentz transformations. Let us tentatively write

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & c g_j \\ N_i/c & \sigma_{ij} \end{pmatrix}\,. \tag{4.20}$$

We shall now show that this object is indeed a tensor since under Lorentz transformations the components transform correctly. In the rest frame S, we have

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{4.21}$$

If $T^{\mu\nu}$ were a tensor, its components in S' would necessarily be given by $T'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$, where $\Lambda^\mu{}_\alpha$ for the standard boost is given in Eq. (2.26). Evaluating the transformation gives

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta} = \begin{pmatrix} \gamma^2\epsilon & -\gamma^2\beta\epsilon & 0 & 0 \\ -\gamma^2\beta\epsilon & \gamma^2\beta^2\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \epsilon' & c g'_x & 0 & 0 \\ N'_x/c & \sigma'_{xx} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{4.22}$$

where in the last equality we used the transformations derived above from physical arguments. Since the components on the right are exactly what we should have in S' from the identification in Eq. (4.20), we conclude that $T^{\mu\nu}$ is indeed a tensor.

Returning to electromagnetism, we have

$$T^{\mu\nu} = \frac{1}{\mu_0}\begin{pmatrix} \frac{1}{2}(\mathbf{E}^2/c^2 + \mathbf{B}^2) & (\mathbf{E}\times\mathbf{B})_j/c \\ (\mathbf{E}\times\mathbf{B})_i/c & \frac{1}{2}(\mathbf{E}/c)^2\delta_{ij} - E_i E_j/c^2 + \frac{1}{2}\mathbf{B}^2\delta_{ij} - B_i B_j \end{pmatrix}\,. \tag{4.23}$$

Since this is quadratic in the fields, we must be able to write it in terms of suitable contractions of the field-strength tensor $F^{\mu\nu}$ with itself. Some straightforward calculation will convince you that

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\alpha}F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)\,. \tag{4.24}$$

This is symmetric as it must be. Noting the invariant $F^{\alpha\beta}F_{\alpha\beta} = 2(\mathbf{B}^2 - \mathbf{E}^2/c^2)$ [see Eq. (2.28)], we have, for example,

$$\begin{aligned}
\mu_0 T^{00} &= F^{0\alpha}F^0{}_\alpha - \tfrac{1}{4}\eta^{00}2(\mathbf{B}^2 - \mathbf{E}^2/c^2) \\
&= F^{0i}F^0{}_i + \tfrac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2/c^2) \\
&= \mathbf{E}^2/c^2 + \tfrac{1}{2}\mathbf{B}^2 - \tfrac{1}{2}\mathbf{E}^2/c^2 \\
&= \tfrac{1}{2}\left(\mathbf{B}^2 + \mathbf{E}^2/c^2\right) = \mu_0\epsilon\,. \qquad (4.25)
\end{aligned}$$

Exercise: Verify that Eq. (4.24) correctly reproduces the expressions for the energy flux and Maxwell stress tensor given in Eqs (4.4) and (4.11) respectively.

Note that $T^{\mu\nu}$ is trace-free since

$$T^\mu{}_\mu = \frac{1}{\mu_0}\left(F^{\mu\alpha}F_{\mu\alpha} - \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) = 0\,, \qquad (4.26)$$

where we used $\eta_{\mu\nu}\eta^{\mu\nu} = 4$.

4.3 Covariant conservation of the stress-energy tensor

Having introduced the stress-energy tensor $T^{\mu\nu}$, we expect to be able to write the conservation laws, Eqs (4.6) and (4.13), in the covariant form

$$\partial_\mu T^{\mu\nu} = \text{some 4-vector linear in the field strength tensor and current 4-vector}\,. \qquad (4.27)$$

Taking the 0-component of $\partial_\mu T^{\mu\nu}$, we have

$$\begin{aligned}
\partial_\mu T^{\mu 0} &= \frac{\partial T^{00}}{\partial(ct)} + \frac{\partial T^{i0}}{\partial x_i} \\
&= \frac{1}{c}\frac{\partial\epsilon}{\partial t} + \frac{1}{c}\boldsymbol{\nabla}\cdot\mathbf{N} \\
&= -\frac{1}{c}\mathbf{J}\cdot\mathbf{E}\,. \qquad (4.28)
\end{aligned}$$

Repeating for the i-component,

$$\begin{aligned}
\partial_\mu T^{\mu i} &= \frac{\partial T^{0i}}{\partial(ct)} + \frac{\partial T^{ji}}{\partial x_j} \\
&= \frac{\partial g_i}{\partial t} + \frac{\partial\sigma_{ji}}{\partial x_j} \\
&= -(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})_i\,, \qquad (4.29)
\end{aligned}$$

so that

$$\partial_\mu T^{\mu\nu} = -\begin{pmatrix} \mathbf{J}\cdot\mathbf{E}/c \\ \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} \end{pmatrix} . \tag{4.30}$$

The right-hand side (which must be a 4-vector) is $-F^\nu{}_\mu J^\mu$, since

$$F^\nu{}_\mu J^\mu = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \rho c \\ J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \mathbf{J}\cdot\mathbf{E}/c \\ \rho E_x + J_y B_z - J_z B_y \\ \rho E_y + J_z B_x - J_x B_z \\ \rho E_z + J_x B_y - J_y B_x \end{pmatrix} . \tag{4.31}$$

It follows that we can combine energy and momentum conservation into the covariant conservation law for the stress-energy tensor:

$$\partial_\mu T^{\mu\nu} = -F^\nu{}_\mu J^\mu . \tag{4.32}$$

The 4-vector on the right of this equation has a simple interpretation in the case of a simple convective flow of charges, each with charge q and number density n_0 in their rest frame. Then, $J^\mu = qn_0 u^\mu$, where u^μ is the 4-velocity, and $F^\nu{}_\mu J^\mu = qn_0 F^\nu{}_\mu u^\mu$ is the product of n_0 and the 4-force on each charge. It follows that, in this case, $F^\nu{}_\mu J^\mu$ is the 4-force per unit rest-frame volume.

We can establish the conservation law, Eq. (4.32), directly from the covariant form of Maxwell's equations. Lowering the ν index, we have

$$\begin{aligned} \mu_0 \partial_\mu T^\mu{}_\nu &= \partial_\mu \left(F^{\mu\alpha} F_{\nu\alpha} - \tfrac{1}{4}\delta^\mu_\nu F_{\alpha\beta} F^{\alpha\beta} \right) \\ &= (\partial_\mu F^{\mu\alpha}) F_{\nu\alpha} + F^{\mu\alpha}\partial_\mu F_{\nu\alpha} - \tfrac{1}{2} F^{\alpha\beta}\partial_\nu F_{\alpha\beta} \\ &= -\mu_0 F_{\nu\alpha} J^\alpha + \tfrac{1}{2}\left(F^{\mu\alpha}\partial_\mu F_{\nu\alpha} + F^{\mu\alpha}\partial_\mu F_{\nu\alpha} - F^{\alpha\beta}\partial_\nu F_{\alpha\beta} \right) , \end{aligned} \tag{4.33}$$

where we used the first of Eq. (2.8) in the last equality. With some relabelling of the dummy indices, the sum of the final three terms can be shown to vanish by virtue of the second of Eq. (2.8),

$$\begin{aligned} F^{\mu\alpha}\partial_\mu F_{\nu\alpha} + F^{\mu\alpha}\partial_\mu F_{\nu\alpha} - F^{\alpha\beta}\partial_\nu F_{\alpha\beta} &= F^{\alpha\beta}\partial_\alpha F_{\nu\beta} + F^{\beta\alpha}\partial_\beta F_{\nu\alpha} - F^{\alpha\beta}\partial_\nu F_{\alpha\beta} \\ &= -F^{\alpha\beta}\left(\partial_\alpha F_{\beta\nu} + \partial_\beta F_{\nu\alpha} + \partial_\nu F_{\alpha\beta} \right) \\ &= 0 , \end{aligned} \tag{4.34}$$

leaving $\partial_\mu T^\mu{}_\nu = -F_{\nu\mu} J^\mu$.

4.4 Stress-energy tensor and Noether's theorem (non-examinable)

In the absence of charges and currents, the electromagnetic stress-energy tensor is conserved: $\partial_\mu T^{\mu\nu} = 0$. Generally, conservation laws can be related to symmetries of the action (*Noether's theorem*). Rather than giving a careful statement of the theorem here, we shall illustrate the idea for the stress-energy tensor.

As a warm-up, consider the non-relativistic Lagrangian for a point particle, Eq. (3.2), in the case that the potential V does not depend on time. The Lagrangian does not have any *explicit* time dependence, and so

$$\frac{dL}{dt} = \dot{\mathbf{x}} \cdot \frac{\partial L}{\partial \mathbf{x}} + \ddot{\mathbf{x}} \cdot \frac{\partial L}{\partial \dot{\mathbf{x}}} \,. \tag{4.35}$$

Using the Euler–Lagrange equation (3.3), we have

$$\begin{aligned} \frac{dL}{dt} &= \dot{\mathbf{x}} \cdot \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) + \ddot{\mathbf{x}} \cdot \frac{\partial L}{\partial \dot{\mathbf{x}}} \\ &= \frac{d}{dt}\left(\dot{\mathbf{x}} \cdot \frac{\partial L}{\partial \dot{\mathbf{x}}}\right) , \end{aligned} \tag{4.36}$$

so that

$$L - \dot{\mathbf{x}} \cdot \frac{\partial L}{\partial \dot{\mathbf{x}}} = \text{const.} \,. \tag{4.37}$$

The combination on the left evaluates to $-m\dot{\mathbf{x}}^2/2 - V$, i.e., (minus) the total energy of the particle. We see that the symmetry of time-translation invariance implies conservation of total energy.

Now consider the free part of the electromagnetic field action, for which the Lagrangian density is

$$\mathcal{L}_{\text{em}} = -\frac{1}{4\mu_0 c} \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) \,. \tag{4.38}$$

This has no explicit dependence on spacetime position, so we are led to consider

$$\partial_\mu \mathcal{L}_{\text{em}} = \partial_\mu A^\alpha \frac{\partial \mathcal{L}_{\text{em}}}{\partial A^\alpha} + \partial_\mu \partial_\nu A^\alpha \frac{\partial \mathcal{L}_{\text{em}}}{\partial\left(\partial_\nu A^\alpha\right)} \,. \tag{4.39}$$

The Euler–Lagrange equation in this case is

$$\frac{\partial \mathcal{L}_{\text{em}}}{\partial A^\alpha} = \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}_{\text{em}}}{\partial\left(\partial_\nu A^\alpha\right)}\right) , \tag{4.40}$$

which gives the expected Maxwell equation $\partial_\mu F^{\mu\nu} = 0$ for vanishing J^ν. Using this in Eq. (4.39), we have

$$\begin{aligned} \partial_\mu \mathcal{L}_{\text{em}} &= \partial_\mu A^\alpha \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}_{\text{em}}}{\partial\left(\partial_\nu A^\alpha\right)}\right) + \partial_\mu \partial_\nu A^\alpha \frac{\partial \mathcal{L}_{\text{em}}}{\partial\left(\partial_\nu A^\alpha\right)} \\ &= \frac{\partial}{\partial x^\nu}\left(\partial_\mu A^\alpha \frac{\partial \mathcal{L}_{\text{em}}}{\partial\left(\partial_\nu A^\alpha\right)}\right) , \end{aligned} \tag{4.41}$$

from which it follows that

$$\frac{\partial}{\partial x^\nu}\left(\delta^\nu_\mu \mathcal{L}_{\rm em} - \partial_\mu A^\alpha \frac{\partial \mathcal{L}_{\rm em}}{\partial\left(\partial_\nu A^\alpha\right)}\right) = 0\,. \tag{4.42}$$

The quantity in brackets is a conserved tensor, but is not quite the electromagnetic stress-energy tensor: using Eq. (4.38), we have

$$\frac{\partial \mathcal{L}_{\rm em}}{\partial\left(\partial_\nu A^\alpha\right)} = -\frac{1}{\mu_0 c} F^\nu{}_\alpha\,, \tag{4.43}$$

so that

$$\delta^\nu_\mu \mathcal{L}_{\rm em} - \partial_\mu A^\alpha \frac{\partial \mathcal{L}_{\rm em}}{\partial\left(\partial_\nu A^\alpha\right)} = \frac{1}{\mu_0 c}\left(\partial_\mu A_\alpha F^{\nu\alpha} - \frac{1}{4}\delta^\nu_\mu F_{\rho\tau}F^{\rho\tau}\right)\,. \tag{4.44}$$

This is neither gauge-invariant or symmetric. It differs (up to a factor of c) from the correct stress-energy tensor in Eq. (4.24) by a term

$$-\frac{1}{\mu_0 c}\partial_\alpha A_\mu F^{\nu\alpha} = -\frac{1}{\mu_0 c}\partial_\alpha\left(A_\mu F^{\nu\alpha}\right)\,, \tag{4.45}$$

where we have used the Maxwell equation $\partial_\alpha F^{\nu\alpha} = 0$. The latter form shows that the difference between the two tensors has vanishing divergence, $\partial_\nu\partial_\alpha(A_\mu F^{\nu\alpha}) = 0$, due to the antisymmetry of the field-strength tensor. We can always add the term (4.45) to the conserved tensor in Eq. (4.44) to form

$$\frac{1}{\mu_0 c}\left(\left(\partial_\mu A_\alpha - \partial_\alpha A_\mu\right)F^{\nu\alpha} - \frac{1}{4}\delta^\nu_\mu F_{\rho\tau}F^{\rho\tau}\right) = \frac{1}{\mu_0 c}\left(F_{\mu\alpha}F^{\nu\alpha} - \frac{1}{4}\delta^\nu_\mu F_{\rho\tau}F^{\rho\tau}\right) = \frac{1}{c}T_\mu{}^\nu\,, \tag{4.46}$$

without spoiling the conservation law. This tensor is now gauge-invariant and symmetric, and is the usual electromagnetic stress-energy tensor.

If the kludge above is not to your taste, we note that there is a direct, but more complicated, way of extracting a symmetric, gauge-invariant stress-energy tensor from the action; see Part III *General Relativity*.

4.5 Symmetry of the stress-energy tensor (non-examinable)

The stress-energy tensor is symmetric since $\mathbf{g} = \mathbf{N}/c^2$ and $\sigma_{ij} = \sigma_{ji}$. We demonstrated above that this was true for electromagnetism but it is true generally. In this section we shall briefly discuss why.

Consider the energy and momentum conservation laws for an isolated system:

$$\frac{\partial \epsilon}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{N} = 0 \tag{4.47}$$

$$\frac{\partial g_i}{\partial t} + \frac{\partial \sigma_{ij}}{\partial x_j} = 0\,. \tag{4.48}$$

The usual 3D angular momentum of the system is

$$\mathbf{L} \equiv \int \mathbf{x} \times \mathbf{g}\, d^3\mathbf{x}\,, \tag{4.49}$$

where the integral extends over all space. Taking the time derivative gives

$$\begin{aligned}
\frac{dL_i}{dt} &= \epsilon_{ijk} \int x_j \frac{\partial g_k}{\partial t}\, d^3\mathbf{x} \\
&= -\epsilon_{ijk} \int x_j \frac{\partial \sigma_{kl}}{\partial x_l}\, d^3\mathbf{x} \\
&= -\epsilon_{ijk} \int \frac{\partial}{\partial x_l} \left(x_j \sigma_{kl}\right)\, d^3\mathbf{x} + \epsilon_{ijk} \int \sigma_{kj}\, d^3\mathbf{x} \\
&= -\epsilon_{ijk} \int x_j \sigma_{kl}\, dS_l + \epsilon_{ijk} \int \sigma_{kj}\, d^3\mathbf{x}\,.
\end{aligned} \tag{4.50}$$

The first term on the right-hand side in the last line is the rate at which angular momentum is entering the (isolated) system from infinity and and must vanish for a localised field configuration. We then see that angular momentum of the isolated system is conserved *provided that* σ_{ij} *is symmetric* so that the last term on the right of Eq. (4.50) vanishes.

Now consider a Lorentz transformation. The stress-energy tensor transforms as

$$T'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}\,. \tag{4.51}$$

Considering the 1-2 component, we have

$$\begin{aligned}
\sigma'_{xy} &= \Lambda^1{}_\alpha \Lambda^2{}_\beta T^{\alpha\beta} \\
&= \Lambda^1{}_0 \Lambda^2{}_2 T^{02} + \Lambda^1{}_1 \Lambda^2{}_2 T^{12} \quad \text{(standard boost)} \\
&= -\gamma\beta T^{02} + \gamma T^{12}\,.
\end{aligned} \tag{4.52}$$

Repeating for the 2-1 component, we have

$$\sigma'_{yx} = -\gamma\beta T^{20} + \gamma T^{21}\,. \tag{4.53}$$

Symmetry of σ_{ij} in all inertial frames therefore requires $T^{i0} = T^{0i}$ and so $\mathbf{g} = \mathbf{N}/c^2$.

A further consequence of the symmetry of the stress-energy tensor follows from considering the dynamics of the "centre of energy" of the system, $\int \epsilon \mathbf{x}\, d^3\mathbf{x} / \int \epsilon\, d^3\mathbf{x}$. Since the total energy is conserved, the dynamics of the centre of energy follow from those of $\int \epsilon \mathbf{x}\, d^3\mathbf{x}$; we have

$$\begin{aligned}\frac{d}{dt}\int \epsilon x_i\, d^3\mathbf{x} &= \int \frac{\partial \epsilon}{\partial t} x_i\, d^3\mathbf{x} \\ &= -\int \frac{\partial N_j}{\partial x_j} x_i\, d^3\mathbf{x} \\ &= -\int \frac{\partial}{\partial x_j}\left(x_i N_j\right)\, d^3\mathbf{x} + \int N_i\, d^3\mathbf{x} \\ &= -\int x_i N_j\, dS_j + \int N_i\, d^3\mathbf{x}\,. \end{aligned} \tag{4.54}$$

The first term on the right-hand side vanishes for a localised field distribution. If $\mathbf{N} = \mathbf{g}c^2$, then $\int \mathbf{N}\, d^3\mathbf{x}$ is constant by the conservation law for $\mathbf{g}$ [Eq. (4.48)], and we see from Eq. (4.54) that the centre of energy moves uniformly. In particular, in the zero momentum frame where $\int \mathbf{g}\, d^3\mathbf{x} = 0$, the centre of energy remains at rest.

The above argument shows that two theorems from mechanics for isolated systems – that the centre of energy moves uniformly and that angular momentum is conserved – are related in relativity. Indeed, one can introduce the object

$$S^{\mu\alpha\beta} \equiv x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} = -S^{\mu\beta\alpha}\,, \tag{4.55}$$

which transforms as a tensor under homogeneous Lorentz transformations. Then, for an isolated system, conservation of the stress-energy tensor implies that

$$\begin{aligned}\partial_\mu S^{\mu\alpha\beta} &= \delta^\alpha_\mu T^{\mu\beta} + x^\alpha \underbrace{\partial_\mu T^{\mu\beta}}_{=0} - \delta^\beta_\mu T^{\mu\alpha} - x^\beta \underbrace{\partial_\mu T^{\mu\alpha}}_{=0} \\ &= T^{\alpha\beta} - T^{\beta\alpha} = 0\,. \end{aligned} \tag{4.56}$$

Integrating this conservation law over space, the $0i$ component returns the centre-of-energy theorem and the ij component returns 3D angular momentum conservation.

4.6 Stress-energy tensor of a plane electromagnetic wave

As an example of computing the stress-energy tensor, we consider electromagnetic plane waves. Maxwell's equations admit wave-like solutions in free space:

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_0 f(\mathbf{k}\cdot\mathbf{x} - \omega t) \\ \mathbf{B} &= \mathbf{B}_0 f(\mathbf{k}\cdot\mathbf{x} - \omega t) \quad \text{with } \omega = c|\mathbf{k}|\,. \end{aligned} \tag{4.57}$$

If the function f is sinusoidal, with period 2π, these are harmonic waves propagating at speed c with wavelength $2\pi/k$ (with $k = |\mathbf{k}|$) and frequency $\omega/(2\pi)$. At any time, the phase of the wave, $\mathbf{k}\cdot\mathbf{x}-\omega t$, is constant on 3D planes perpendicular to the wavevector $\mathbf{k}$. Since $\boldsymbol{\nabla}\cdot\mathbf{E}=0$ in free space, and $\boldsymbol{\nabla}\cdot\mathbf{B}=0$, we must have

$$\mathbf{k}\cdot\mathbf{E}_0=0 \quad \text{and} \quad \mathbf{k}\cdot\mathbf{B}_0=0\,. \tag{4.58}$$

Furthermore, from $\boldsymbol{\nabla}\times\mathbf{E}=-\partial\mathbf{B}/\partial t$ we have

$$\mathbf{k}\times\mathbf{E}_0=\omega\mathbf{B}_0 \quad \Rightarrow \quad \hat{\mathbf{k}}\times\mathbf{E}_0=c\mathbf{B}_0\,, \tag{4.59}$$

where $\hat{\mathbf{k}}$ is a unit vector in the direction of $\mathbf{k}$, and so $c|\mathbf{B}_0|=|\mathbf{E}_0|$. It follows that $\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{k}$ form a right-handed triad of vectors.

We can write the fields in covariant form as follows. Phase differences between two spacetime events must be Lorentz invariant so we can introduce the 4-wavevector

$$k^\mu=(\omega/c,\mathbf{k})\,. \tag{4.60}$$

This is a null 4-vector, $\eta_{\mu\nu}k^\mu k^\nu=0$, since $\omega=ck$. The phase can then be written covariantly as $\mathbf{k}\cdot\mathbf{x}-\omega t=k_\mu x^\mu$. Specialising to $\mathbf{k}$ along the x-direction of some inertial frame, and $\mathbf{E}_0$ along y (so that $\mathbf{B}_0$ is along z), the components of the field-strength tensor in that frame are

$$F^{\mu\nu}=B_0 f(k_\mu x^\mu)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{4.61}$$

where $B_0=|\mathbf{B}_0|$. Both invariants $F^{\mu\nu}F_{\mu\nu}$ and ${}^\star F^{\mu\nu}F_{\mu\nu}$ vanish since $\mathbf{E}\cdot\mathbf{B}=0$ and $|\mathbf{E}|=c|\mathbf{B}|$.

The stress-energy tensor, Eq. (4.23), reduces to $T^{\mu\nu}=F^{\mu\alpha}F^{\nu}{}_{\alpha}/\mu_0$ since $F^{\mu\nu}F_{\mu\nu}=0$. It follows that

$$\begin{aligned} T^{\mu\nu} &= \frac{B_0^2 f^2(k_\mu x^\mu)}{\mu_0}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \frac{B_0^2 f^2(k_\mu x^\mu)}{\mu_0}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \end{aligned} \tag{4.62}$$

We can read off the energy density

$$\epsilon=B_0^2 f^2(k_\mu x^\mu)/\mu_0\,, \tag{4.63}$$

which receives equal contributions from the electric and magnetic fields. The Poynting vector is along the x-direction, with

$$N_x = cB_0^2 f^2(k_\mu x^\mu)/\mu_0 \,. \tag{4.64}$$

Finally, the momentum flux has only one non-zero component:

$$\sigma_{xx} = B_0^2 f^2(k_\mu x^\mu)/\mu_0 \,. \tag{4.65}$$

Note that

$$\sigma_{xx} = N_x/c = cg_x = \epsilon \,. \tag{4.66}$$

These results are consistent with the photon picture in which photons with energy $E = pc$ (where p is the magnitude of their momentum) propagate at speed c along the x-direction. The energy density is then c times the momentum density, and the momentum flux is c times the momentum density[3].

Exercise: Consider reflection of a plane electromagnetic wave, propagating along the x-direction, from a perfect conductor whose plane surface is at $x = 0$. The conductivity is so high that the fields decay very rapidly inside the conductor and we can therefore approximate the fields as being zero inside. The **E** field is tangent to the surface and, since the tangential component is continuous, the **E** field must be zero at the surface of the conductor. The **B** field is also tangential but is now discontinuous at the surface with the discontinuity driven by a surface current flowing in a thin layer on the surface.

The total electric and magnetic fields (sum of incident and reflected waves) outside the conductor ($x < 0$) are therefore standing waves of the form

$$\begin{aligned} \mathbf{E}_{\rm tot} &= B_0 c\,[f(kx - \omega t) - f(-kx - \omega t)]\,\hat{\mathbf{y}} \\ \mathbf{B}_{\rm tot} &= B_0\,[f(kx - \omega t) + f(-kx - \omega t)]\,\hat{\mathbf{z}}\,. \end{aligned} \tag{4.67}$$

Show that the energy density is

$$\epsilon = \frac{B_0^2}{\mu_0}\left[f^2(kx - \omega t) + f^2(-kx - \omega t)\right] \,, \tag{4.68}$$

which is the sum of the energy density in the incident and reflected waves. For harmonic waves, the time-averaged energy density is the same at all x. Show also that the Poynting vector has a single non-zero component

$$N_x = \frac{B_0^2 c}{\mu_0}\left[f^2(kx - \omega t) - f^2(-kx - \omega t)\right] \,, \tag{4.69}$$

which, again, is the sum of the incident and reflected energy fluxes. The Poynting vector vanishes at the surface of the conductor at all times, so no energy enters the conductor. This

[3]The momentum carried through an area dS in the y-z plane in time dt is the total momentum of those photons within a volume $cdtdS$, i.e., $\sigma_{xx}dSdt = g_x cdtdS$.

is still consistent with there being a surface current since the rate of energy dissipation by this current scales as the inverse of the conductivity and so vanishes in the limit of infinite conductivity assumed here. At other locations, the time-averaged Poynting vector vanishes for harmonic waves. Finally, show that the Maxwell stress-tensor is diagonal with

$$\begin{aligned} \sigma_{xx} &= \frac{B_0^2}{\mu_0}\left[f^2(kx-\omega t)+f^2(-kx-\omega t)\right] \\ \sigma_{yy} &= \frac{2B_0^2}{\mu_0}f(kx-\omega t)f(-kx-\omega t)\,, \end{aligned} \tag{4.70}$$

and $\sigma_{zz} = -\sigma_{yy}$. The $\sigma_{xx} = \epsilon$ is the sum of the incident and reflected momentum fluxes. However, the σ_{yy} and σ_{zz} components arise purely from interference of the incident and reflected waves. Since σ_{yy} and σ_{zz} only vary in the x-direction, the non-zero values are still consistent with the momentum density of the field and the conservation law (4.48).

There is a force exerted on the conductor ("radiation pressure") in the x-direction given per area by

$$\sigma_{xx}|_{x=0} = \frac{2B_0^2}{\mu_0}f^2(-\omega t)\,. \tag{4.71}$$

This follows from the integral form of the momentum conservation law, Eq. (4.14), noting the momentum density is zero inside the conductor. The *incident* momentum density at $x=0$ is $B_0^2 f^2(-\omega t)/(\mu_0 c)$, and so the radiation pressure is $2c$ times the incident momentum density. What is the physical origin of this force? It is the Lorentz force of the $\mathbf{B}$ field at the surface on the surface current distribution. To see that this is consistent, note that the surface current $\mathbf{J}_s$ (defined such that the current through a given line element dl lying in the surface and perpendicular to $\mathbf{J}_s$ is $|\mathbf{J}_s|dl$) needed to maintain the discontinuity in $\mathbf{B}$ is along the x-direction with $\mu_0 J_s = B_y|_{x=0}$. The force per area on this current is $J_s B_y|_{x=0}/2$ along the x-direction. The factor of 1/2 here accounts for the average value of the magnetic field across the depth of the current distribution. Using $B_y|_{x=0} = 2B_0(-\omega t)$, we see that the Lorentz force per area is exactly σ_{xx} at the surface of the conductor, as required.

4.6.1 Radiation pressure of a photon "gas"

We can obtain an *isotropic* distribution of radiation by forming a random superposition of incoherent plane waves propagating in all directions with equal intensity. (Incoherent waves have no lasting phase relation amongst themselves, e.g., the radiation emitted by two separated thermal sources.) Such radiation is the classical analogue of a *photon gas*. Isotropy requires the (time-averaged) Poynting vector to vanish and that the Maxwell stress tensor $\sigma_{ij} = p\delta_{ij}$, where p is the pressure. The averaged stress-energy tensor is then diagonal with

$$\langle T^{\mu\nu}\rangle = \mathrm{diag}(\epsilon, p, p, p)\,. \tag{4.72}$$

The electromagnetic stress energy tensor is necessarily trace-free;

$$T^{\mu}{}_{\mu} = 0 \quad \Rightarrow \quad p = \epsilon/3\,. \tag{4.73}$$

We see that for a photon gas the pressure is one-third of the energy density irrespective of the way that the energy is distributed over frequency (i.e., the radiation does not have to have a blackbody spectrum).

Part II

Radiation of electromagnetic waves

In this part we consider the *production* of electromagnetic radiation from time-dependent charge distributions. We first consider radiation from charge and current densities involving non-relativistic motions, as, for example, in radiation from a radio antenna. As we shall see, it is the acceleration of the charges that determines the radiation field. As an application of these ideas, we then briefly consider *scattering* of electromagnetic radiation. To describe radiation from charges moving relativistically requires an alternative treatment, which we subsequently develop. This theory underlies extreme phenomena such as the *synchrotron radiation* emitted from charges circling relativistically in magnetic fields.

5 Retarded potentials of a time-dependent charge distribution

We begin by deriving the 4-potential generated causally by an arbitrary 4-current J^μ. To do this, we combine $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with the Maxwell equation

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \,, \tag{5.1}$$

to obtain

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = -\mu_0 J^\mu \,, \tag{5.2}$$

where the operator

$$\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{5.3}$$

is the wave operator. It is very convenient to make a gauge choice (the *Lorenz gauge*) such that $\partial_\mu A^\mu = 0$. This is clearly a Lorentz-invariant statement, which has the benefit of retaining covariance of the subsequent gauge-fixed equations under Lorentz transformations. It is always possible to make this gauge choice: starting in some general gauge we make a transformation $A^\mu \to A^\mu + \partial^\mu\chi$ with χ chosen to satisfy $\Box\chi = -\partial_\mu A^\mu$. As we show below, we can always solve this equation to determine a suitable χ. In the Lorenz gauge, Eq. (5.2) simplifies to

$$\Box A^\mu = -\mu_0 J^\mu \quad \text{or} \quad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A^\mu = -\mu_0 J^\mu \,, \tag{5.4}$$

which is a sourced wave equation.

We solve Eq. (5.4) by taking a Fourier transform in time, writing

$$A^\mu(t, \mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{A}^\mu(\omega, \mathbf{x}) e^{i\omega t}\, d\omega\,, \tag{5.5}$$

and similarly for J^μ. The sourced wave equation then becomes

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{A}^\mu = -\mu_0 \tilde{J}^\mu\,. \tag{5.6}$$

This is a sourced Helmholtz equation with $k^2 = \omega^2/c^2$. As for Poisson's equation in electrostatics, we can solve this with a Green's function $G(\mathbf{x};\mathbf{x}')$ such that

$$\left(\nabla^2 + k^2\right) G(\mathbf{x};\mathbf{x}') = \delta^{(3)}(\mathbf{x} - \mathbf{x}')\,, \tag{5.7}$$

so that

$$\tilde{A}^\mu(\omega, \mathbf{x}) = -\mu_0 \int G(\mathbf{x};\mathbf{x}') \tilde{J}^\mu(\omega, \mathbf{x}')\, d^3\mathbf{x}'\,. \tag{5.8}$$

To describe the fields produced by localised sources, we require the Green's function to tend to zero as $|\mathbf{x}| \to 0$. Homogeneity and isotropy require that $G(\mathbf{x};\mathbf{x}')$ is a function of $|\mathbf{x} - \mathbf{x}'|$ alone. Taking $\mathbf{x}' = 0$ and $r = |\mathbf{x}|$, away from $r = 0$ we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) + k^2 G = 0\,. \tag{5.9}$$

The solution that tends to zero as $r \to \infty$ is $G = -Ae^{-ikr}/r$. (The reason for choosing e^{-ikr} rather than e^{ikr} is to do with causality and will be explained shortly.) We fix the normalisation A by integrating Eq. (5.7) over a sphere of radius ϵ centred on $\mathbf{x}'$, and taking the limit as $\epsilon \to 0$. The integral of the $k^2 G$ term goes like ϵ^3/ϵ and so vanishes in the limit, leaving

$$\begin{aligned} &\lim_{r\to 0} 4\pi r^2 \partial G/\partial r = 1 \\ \Rightarrow \quad &\lim_{r\to 0} 4\pi r^2 A \left(\frac{e^{-ikr}}{r^2} + ik\frac{e^{-ikr}}{r}\right) = 1\,, \end{aligned} \tag{5.10}$$

so that $A = 1/(4\pi)$ and

$$G(\mathbf{x};\mathbf{x}') = -\frac{1}{4\pi}\frac{e^{-i\omega|\mathbf{x}-\mathbf{x}'|/c}}{|\mathbf{x}-\mathbf{x}'|}\,. \tag{5.11}$$

Inserting this Green's function into Eq. (5.8) and taking the inverse Fourier transform, gives

$$A^\mu(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \int d^3\mathbf{x}'\, \frac{e^{i\omega(t - |\mathbf{x}-\mathbf{x}'|/c)}}{|\mathbf{x}-\mathbf{x}'|} \tilde{J}^\mu(\omega, \mathbf{x}')\,. \tag{5.12}$$

The integral over ω is the inverse Fourier transform of $\tilde{J}^\mu(\omega, \mathbf{x}')$ and returns $J^\mu(t_{\text{ret}}, \mathbf{x}')$ where the *retarded time* $t_{\text{ret}} \equiv t - |\mathbf{x} - \mathbf{x}'|/c$. This gives our final result for the 4-potential from an isolated time-dependent charge distribution:

$$A^\mu(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{J^\mu(t_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3\mathbf{x}' \, . \tag{5.13}$$

Equation (5.13) has a simple physical interpretation. In components, it becomes

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(t_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3\mathbf{x}' \, , \qquad \mathbf{J}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(t_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3\mathbf{x}' \, . \tag{5.14}$$

If we were in a static situation, $\rho(t_{\text{ret}}, \mathbf{x}') = \rho(t, \mathbf{x}')$, and similarly for the current, and these equations would become the usual potentials for electrostatics and magnetostatics (in the Coulomb gauge, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$). The only difference for time-dependent charge distributions is that the sources have to be evaluated at the retarded time. Since the event at t_{ret} and $\mathbf{x}'$ is on the *past lightcone* of the event at t and $\mathbf{x}$, the field at any event x^μ is generated by the superposition of the fields due to currents at earlier times on the past lightcone of x^μ. This reflects causality in that electromagnetic disturbances propagate at the speed of light and that only earlier events can influence the field at any spacetime point[4].

The potential A^μ of Eq. (5.13) is called the *retarded potential.* We now see why we had to choose the e^{-ikr}/r form for the Green's function – had we chosen e^{ikr}/r we would have obtained the *advanced potential* for which the currents at $t + |\mathbf{x} - \mathbf{x}'|/c$ and $\mathbf{x}'$ influence the field at t and $\mathbf{x}$, i.e., acausal propagation.

It is not obvious that Eq. (5.13) is covariant under Lorentz transformations, since we are integrating over space alone. To see that it is covariant, we rewrite it as a spacetime integral as follows:

$$A^\mu(x) = \frac{\mu_0}{4\pi} \int J^\mu(y) \frac{\delta(y^0 - x^0 + |\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} \, d^4y \, . \tag{5.15}$$

This looks more covariant since it involves the Lorentz-invariant measure d^4y. However, we still have the issue of the delta-function. To see that this is Lorentz invariant, consider

$$\delta\left(\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)\right) \Theta(x^0 - y^0) \, , \tag{5.16}$$

where Θ is the Heaviside (unit-step) function, which equals unity if its argument is greater than zero and vanishes otherwise. The product of the two terms in Eq. (5.16) is therefore Lorentz invariant since the delta-function is non-zero only for null-separated x^μ and y^μ, and the temporal ordering of such events is Lorentz invariant. We now show

[4]Recall that the temporal ordering of timelike and null-separated events is Lorentz invariant

that Eq. (5.16) is twice the term $\delta(y^0 - x^0 + |\mathbf{x}-\mathbf{y}|)/|\mathbf{x}-\mathbf{y}|$ in question in Eq. (5.15). We do this by regarding the argument of the delta-function as a function of y^0, noting that

$$\begin{aligned}\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) &= |\mathbf{x}-\mathbf{y}|^2 - (x^0 - y^0)^2 \\ &= (|\mathbf{x}-\mathbf{y}| + x^0 - y^0)(|\mathbf{x}-\mathbf{y}| - x^0 + y^0)\,,\end{aligned} \tag{5.17}$$

and use the result in Eq. (3.55) for delta-functions with functions as their arguments. Summing over both roots of the argument, this gives

$$\begin{aligned}\delta\left(\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)\right) &= \frac{\delta(y^0 - x^0 + |\mathbf{x}-\mathbf{y}|)}{2|y^0 - x^0|} + \frac{\delta(y^0 - x^0 - |\mathbf{x}-\mathbf{y}|)}{2|y^0 - x^0|} \\ &= \frac{\delta(y^0 - x^0 + |\mathbf{x}-\mathbf{y}|)}{2|\mathbf{x}-\mathbf{y}|} + \frac{\delta(y^0 - x^0 - |\mathbf{x}-\mathbf{y}|)}{2|\mathbf{x}-\mathbf{y}|}\,.\end{aligned} \tag{5.18}$$

The Heaviside function in Eq. (5.16) selects only the first term on the right, showing that

$$\delta\left(\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)\right)\Theta(x^0 - y^0) = \frac{\delta(y^0 - x^0 + |\mathbf{x}-\mathbf{y}|)}{2|\mathbf{x}-\mathbf{y}|}\,, \tag{5.19}$$

as required. Equation (5.15) therefore involves a Lorentz-invariant integration of the 4-vector current over the past lightcone of the event x^μ, and so is properly covariant.

Exercise: Noting that the gauge condition $\partial_\mu A^\mu = 0$ takes the form $i\omega\tilde{A}^0/c + \boldsymbol{\nabla}\cdot\tilde{\mathbf{A}} = 0$ after Fourier transforming in time, verify that the right-hand side of Eq. (5.8) satisfies the gauge condition by virtue of charge conservation.

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6 Dipole radiation

As a first application of the retarded potential derived above, we consider radiation from some charge distribution in which all motions are non-relativistic (in some inertial frame). We shall be interested in the fields at large distances compared to the spatial extent of the charge distribution.

Specifically, consider the case where all charges are localised within some region of size a, and place the origin of spatial coordinates within this volume. To calculate the 4-potential at $\mathbf{x}$, where $r = |\mathbf{x}| \gg a$, we Taylor expand $|\mathbf{x}-\mathbf{x}'|$ in Eq. (5.13) as

$$\begin{aligned}|\mathbf{x}-\mathbf{x}'| &= r - \frac{\mathbf{x}\cdot\mathbf{x}'}{r} + \cdots \\ &= r - \hat{\mathbf{x}}\cdot\mathbf{x}' + \cdots\,,\end{aligned} \tag{6.1}$$

where $\hat{\mathbf{x}}$ is a unit vector in the direction of $\mathbf{x}$. Note that the second term on the right is smaller than the first by $O(a/r)$. Substituting into Eq. (5.13), we have

$$A^\mu(t,\mathbf{x}) = \frac{\mu_0}{4\pi r}\int J^\mu(t_{\text{ret}},\mathbf{x}')\left(1+\frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{r}+\cdots\right)d^3\mathbf{x}'\,, \tag{6.2}$$

where $t_{\text{ret}} = t - r/c + \hat{\mathbf{x}}\cdot\mathbf{x}'/c + \cdots$. At $r \gg a$, we can safely neglect the $O(a/r)$ corrections in the round brackets in Eq. (6.2). However, the treatment of the dependence of t_{ret} on $\mathbf{x}'$ depends on the rate at which the source varies. For an oscillating charge distribution, with angular frequency ω, if $a/c \ll 1/\omega$ then the variation in t_{ret} over the source is short compared to the period and can be handled perturbatively via a Taylor expansion of J^μ in time. Note that the typical velocity of the charges is $a\omega$ and so we require this to be much less than c for an expansion to be valid. Equivalently, since the fields radiated by a source oscillating at ω oscillate at the same frequency, the associated wavelength λ must be large compared to a.

For sufficiently slowly-varying sources, we can neglect the variation of t_{ret} across the source. The leading-order contribution to the magnetic vector potential is then

$$\mathbf{A}(t,\mathbf{x}) = \frac{\mu_0}{4\pi r}\int \mathbf{J}(t-r/c,\mathbf{x}')\,d^3\mathbf{x}'\,. \tag{6.3}$$

We can express the right-hand side in terms of the time derivative of the *electric dipole moment* $\mathbf{p}$ of the charge distribution,[5] where, recall from IB *Electromagnetism*,

$$\mathbf{p}(t) \equiv \int \rho(t,\mathbf{x})\mathbf{x}\,d^3\mathbf{x}\,. \tag{6.4}$$

Charge conservation, $\partial\rho/\partial t + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$, implies that

$$\begin{aligned}\dot{\mathbf{p}} &= \frac{d}{dt}\int \rho(t,\mathbf{x})\mathbf{x}\,d^3\mathbf{x}\\ &= \int \frac{\partial\rho}{\partial t}\mathbf{x}\,d^3\mathbf{x}\\ &= -\int (\boldsymbol{\nabla}\cdot\mathbf{J})\mathbf{x}\,d^3\mathbf{x}\,,\end{aligned} \tag{6.5}$$

so that

$$\begin{aligned}\dot{p}_i &= -\int \frac{\partial J_j}{\partial x_j}x_i\,d^3\mathbf{x}\\ &= \int J_j\frac{\partial x_i}{\partial x_j}\,d^3\mathbf{x}\\ &= \int J_i\,d^3\mathbf{x}\,,\end{aligned} \tag{6.6}$$

[5] We are using the same symbol for the electric dipole moment and the 3-momentum. This should not cause any confusion, with the meaning being clear from the context.

where the second equality follows from the generalised divergence theorem and noting that the surface term vanishes for a localised current distribution. We therefore have

$$\mathbf{A}(t,\mathbf{x}) \approx \frac{\mu_0}{4\pi r}\dot{\mathbf{p}}(t-r/c)\,. \tag{6.7}$$

This is called the *electric dipole* approximation. For a source oscillating at frequency ω, we have $\mathbf{A} \sim e^{-i\omega(t-r/c)}/r$, i.e., outgoing spherical waves.

We can calculate the magnetic field from $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. We have to be careful to remember that $t-r/c$ depends on $\mathbf{x}$, so that for some function $f(t-r/c)$,

$$\begin{aligned}\frac{\partial f(t-r/c)}{\partial x_i} &= -\frac{1}{c}f'(t-r/c)\frac{\partial r}{\partial x_i}\\ &= -\frac{x_i}{rc}f'(t-r/c)\,.\end{aligned} \tag{6.8}$$

It follows that

$$\begin{aligned}\mathbf{B}(t,\mathbf{x}) &= -\frac{\mu_0}{4\pi r^2}(\boldsymbol{\nabla}r)\times\dot{\mathbf{p}}(t-r/c) - \frac{\mu_0}{4\pi rc}(\boldsymbol{\nabla}r)\times\ddot{\mathbf{P}}(t-r/c)\\ &= -\frac{\mu_0}{4\pi r^2}\hat{\mathbf{x}}\times\dot{\mathbf{p}}(t-r/c) - \frac{\mu_0}{4\pi rc}\hat{\mathbf{x}}\times\ddot{\mathbf{p}}(t-r/c)\,.\end{aligned} \tag{6.9}$$

Note that the first contribution to the field varies as $1/r^2$ while the second goes like $1/r$. For a source oscillating at ω, the relative size of the $1/r$ part compared to the $1/r^2$ part is $O(r/\lambda)$. We define the far-field (sometimes known as the radiation zone or the wave zone) as the asymptotic region $r \gg \lambda$. In the far-field, the $1/r$ term dominates the magnetic field, and describes outgoing spherical waves of radiation:

$$\mathbf{B}(t,\mathbf{x}) = -\frac{\mu_0}{4\pi rc}\hat{\mathbf{x}}\times\ddot{\mathbf{p}}(t-r/c) \qquad (r \gg \lambda)\,. \tag{6.10}$$

Note that the $\mathbf{B}$ field is perpendicular to $\hat{\mathbf{x}}$ and to $\ddot{\mathbf{p}}$, and that it is $\ddot{\mathbf{p}}$ that determines the radiation field.

For a set of charges $\{q_i\}$ located at $\{\mathbf{x}_i\}$, we have $\mathbf{p} = \sum_i q_i\mathbf{x}_i$ and so

$$\ddot{\mathbf{p}} = \sum_i q_i\ddot{\mathbf{x}}_i\,. \tag{6.11}$$

This shows that it is *acceleration* of the charges that generates the dipole radiation field.

We have to work a little harder for the electric potential since, with the same order of approximation as we used above for $\mathbf{A}$, the electric potential ϕ involves $\int \rho(t - r/c,\mathbf{x}')\,d^3\mathbf{x}'$, i.e., the total charge Q at time $t-r/c$. Since Q is independent of time for an isolated charge distribution, the potential is also time-independent to this level of approximation (it is just the usual asymptotic Coulomb field). We therefore have to work at higher order in a/λ to find the part of ϕ that describes radiation[6]. It is straightforward to do so, but here we shall follow the simpler approach of using the Lorenz gauge condition to find ϕ.

If we use Eq. (6.7) in $\boldsymbol{\nabla}\cdot\mathbf{A}+c^{-2}\partial\phi/\partial t=0$, we find

$$\frac{\partial\phi}{\partial t}=\frac{\mu_0 c^2}{4\pi r^2}\hat{\mathbf{x}}\cdot\dot{\mathbf{p}}(t-r/c)+\frac{\mu_0}{4\pi rc}\hat{\mathbf{x}}\cdot\ddot{\mathbf{p}}(t-r/c)\,, \tag{6.12}$$

which integrates to give

$$\phi(t,\mathbf{x})=\frac{Q}{4\pi\epsilon_0 r}+\frac{1}{4\pi\epsilon_0 r^2}\hat{\mathbf{x}}\cdot\mathbf{p}(t-r/c)+\frac{1}{4\pi\epsilon_0 rc}\hat{\mathbf{x}}\cdot\dot{\mathbf{p}}(t-r/c)\,. \tag{6.13}$$

Here, we have included the Coulomb term from the constant charge Q, which appears as an integration constant. The second term is the usual dipole potential that would be generated by a static charge distribution, but here it depends on the electric dipole moment at the earlier time $t-r/c$. Typically, this term is smaller than the Coulomb term by $O(a/r)$. It is the third term that describes the radiation component of the potential. It varies as $1/r$ and is typically $O(r/\lambda)$ times the second term, and so dominates in the far-field.

We can form the electric field from $\mathbf{E}=-\partial\mathbf{A}/\partial t-\boldsymbol{\nabla}\phi$. Equations (6.7) and (6.13) give terms varying as $1/r$, $1/r^2$ and $1/r^3$. The part going as $1/r$ dominates in the far-field:

$$\begin{aligned}\mathbf{E}(t,\mathbf{x})&=-\frac{\mu_0}{4\pi r}\left[\ddot{\mathbf{p}}(t-r/c)-\hat{\mathbf{x}}\hat{\mathbf{x}}\cdot\ddot{\mathbf{p}}(t-r/c)\right]\qquad(r\gg\lambda)\\&=\frac{\mu_0}{4\pi r}\hat{\mathbf{x}}\times\left[\hat{\mathbf{x}}\times\ddot{\mathbf{p}}(t-r/c)\right].\end{aligned} \tag{6.14}$$

It originates from the time derivative of $\dot{\mathbf{p}}$ in $\mathbf{A}$ (Eq. 6.7) and the spatial derivative of $\dot{\mathbf{p}}(t-r/c)$ in ϕ (Eq. 6.13). This expression for the electric field in the far-field can also be obtained directly from the magnetic field with $\boldsymbol{\nabla}\times\mathbf{B}=c^{-2}\partial\mathbf{E}/\partial t$. The electric field is perpendicular to $\hat{\mathbf{x}}$ and lies in the plane formed from $\hat{\mathbf{x}}$ and $\ddot{\mathbf{p}}$.

We see that in the far-field,

$$\mathbf{E}=-c\hat{\mathbf{x}}\times\mathbf{B}\qquad(r\gg\lambda)\,, \tag{6.15}$$

so that $|\mathbf{E}|=c|\mathbf{B}|$, and $\mathbf{E}\cdot\mathbf{B}=0$. This behaviour is just like for a plane wave with wavevector along $\hat{\mathbf{x}}$. This makes sense since in the far-field, the radius r is so large compared to the wavelength λ that on the scale of λ the spherical wavefronts are effectively planar.

[6]The 3-current $\mathbf{J}\sim\rho\mathbf{v}$ is higher-order in $v/c\sim a/\lambda$ than ρ, which is why we can apparently work to lower order in a/λ when dealing with the vector potential.

6.1 Power radiated

Energy is carried off to spatial infinity from the source in the form of electromagnetic radiation. We can determine the rate at which energy is radiated by integrating the Poynting vector over a spherical surface at very large radius. Working in the far-field, we have

$$\begin{aligned}\mathbf{N} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \\ &= \frac{c}{\mu_0}\mathbf{B}\times(\hat{\mathbf{x}}\times\mathbf{B}) \\ &= \frac{c}{\mu_0}|\mathbf{B}|^2\hat{\mathbf{x}} \\ &= \frac{\mu_0}{(4\pi r)^2 c}|\hat{\mathbf{x}}\times\ddot{\mathbf{p}}|^2\hat{\mathbf{x}}\,. \end{aligned} \tag{6.16}$$

This is radial, as expected. The magnitude of the Poynting vector varies as $1/r^2$, so the integral over a spherical surface of radius r is independent of r. If $\ddot{\mathbf{p}}$ is along the z-axis, $|\hat{\mathbf{x}}\times\ddot{\mathbf{p}}| = \sin\theta|\ddot{\mathbf{p}}|$, where θ is the angle that $\hat{\mathbf{x}}$ makes with the z-axis. In this case,

$$\mathbf{N} = \frac{\mu_0}{(4\pi r)^2 c}|\ddot{\mathbf{p}}|^2\sin^2\theta\,\hat{\mathbf{x}}\,. \tag{6.17}$$

The total radiated power then follows from

$$\begin{aligned}\int \mathbf{N}\cdot d\mathbf{S} &= \frac{\mu_0}{(4\pi)^2 c}|\ddot{\mathbf{p}}|^2\int_{-1}^{1}\sin^2\theta\,d\cos\theta\int_0^{2\pi}d\phi \\ &= \frac{\mu_0}{6\pi c}|\ddot{\mathbf{p}}|^2\,. \end{aligned} \tag{6.18}$$

The power radiated per solid angle is $|\mathbf{N}|r^2$ and varies as $\sin^2\theta$. The radiation is maximal in the plane perpendicular to $\ddot{\mathbf{p}}$ and there is no radiation along the direction of $\ddot{\mathbf{p}}$. The radiation is linearly polarized, with the electric field in the plane containing $\hat{\mathbf{x}}$ and $\ddot{\mathbf{p}}$. The magnetic field is perpendicular to $\mathbf{E}$.

6.2 Beyond the electric-dipole approximation

For many applications involving non-relativistic motions, the electric-dipole approximation is sufficient. However, there are situations where this leading-order contribution vanishes, for example if $\mathbf{p}$ itself vanishes or is constant in time. In such situations it is necessary to expand the source $J^\mu(t_{\mathrm{ret}},\mathbf{x}')$ to higher order in a/λ. Retaining the next-order terms, we have

$$\mathbf{A}(t,\mathbf{x}) \approx \frac{\mu_0}{4\pi r}\dot{\mathbf{p}}(t-r/c) + \frac{\mu_0}{4\pi r c}\int \dot{\mathbf{J}}(t-r/c,\mathbf{x}')\hat{\mathbf{x}}\cdot\mathbf{x}'\,d^3\mathbf{x}'\,. \tag{6.19}$$

The leading correction to the electric-dipole approximation involves the time derivative of the integral $\int J_i x_j \, d^3\mathbf{x}$. We can express this in terms of multipole moments of the charge and current distribution as follows. Since $\partial\rho/\partial t + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$, we have

$$\int_S (x_j x_k \dots x_l) \left(\boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right) d^3\mathbf{x} = 0 \,, \tag{6.20}$$

so that

$$\begin{aligned} \frac{d}{dt} \int \rho x_j x_k \cdots x_l \, d^3\mathbf{x} &= - \int (x_j x_k \dots x_l) \boldsymbol{\nabla} \cdot \mathbf{J} \, d^3\mathbf{x} \\ &= \int \mathbf{J} \cdot \boldsymbol{\nabla} \left(x_j x_k \dots x_l \right) \, d^3\mathbf{x} \\ &= \int \left[(J_j x_k \dots x_l) + \cdots + (x_j x_k \dots J_l) \right] \, d^3\mathbf{x} \,, \end{aligned} \tag{6.21}$$

where we used the generalisation of the divergence theorem in going to the second line and dropped the surface term since it vanishes for a localised charge distribution. A special case of this result that is useful for our purposes here is

$$\int (J_i x_j + J_j x_i) \, d^3\mathbf{x} = \frac{d}{dt} \int \rho x_i x_j \, d^3\mathbf{x} \,. \tag{6.22}$$

We can now write

$$\int J_i x_j \, d^3\mathbf{x} = \frac{1}{2} \int J_i x_j \, d^3\mathbf{x} + \frac{1}{2} \left(\frac{d}{dt} \int \rho x_i x_j \, d^3\mathbf{x} - \int J_j x_i \, d^3\mathbf{x} \right) \,, \tag{6.23}$$

so that

$$\int J_i(t, \mathbf{x}') \hat{\mathbf{x}} \cdot \mathbf{x}' \, d^3\mathbf{x}' = \frac{1}{2} \left[\left(\int \mathbf{x}' \times \mathbf{J}(t, \mathbf{x}') \, d^3\mathbf{x}' \right) \times \hat{\mathbf{x}} \right]_i + \frac{1}{6} \frac{d}{dt} \left(\int 3\rho(t, \mathbf{x}') x'_i x'_j \right) \hat{x}_j \,. \tag{6.24}$$

The first term on the right involves the *magnetic dipole moment*,

$$\mathbf{m}(t) \equiv \frac{1}{2} \int \mathbf{x}' \times \mathbf{J}(t, \mathbf{x}') \, d^3\mathbf{x}' \,. \tag{6.25}$$

The second term on the right is related to the time derivative of the trace-free *electric-quadrupole moment*,

$$Q_{ij}(t) \equiv \int \rho(t, \mathbf{x}') (3 x'_i x'_j - \delta_{ij} |\mathbf{x}'|^2) \, d^3\mathbf{x}' \,. \tag{6.26}$$

The trace term does not appear in Eq. (6.24), but it would only add a radial component (i.e., proportional to $\hat{\mathbf{x}}$) to $\mathbf{A}$ and so would not contribute to the electric and magnetic fields in the far-field (see the stucture of the second, dominant, term in Eq. 6.9).

Putting these results together, we can take

$$\mathbf{A}(t,\mathbf{x}) \approx \frac{\mu_0}{4\pi r}\dot{\mathbf{p}}(t-r/c) + \frac{\mu_0}{4\pi rc}\dot{\mathbf{m}}(t-r/c)\times\hat{\mathbf{x}} + \frac{1}{6}\frac{\mu_0}{4\pi rc}\ddot{\mathsf{Q}}(t-r/c)\hat{\mathbf{x}}\,, \qquad (6.27)$$

where $\mathsf{Q}\hat{\mathbf{x}}$ denotes the contraction of the tensor Q_{ij} with $\hat{x}_j$. The magnetic-dipole and electric-quadrupole terms are of the same order, but a/λ down compared to the electric-dipole term. The magnetic field in the far-field follows from $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$; keeping only the dominant terms that go as $1/r$, we have

$$\mathbf{B}(t,\mathbf{x}) \approx \frac{\mu_0}{4\pi rc}\ddot{\mathbf{p}}(t-r/c)\times\hat{\mathbf{x}} + \frac{\mu_0}{4\pi rc^2}\left[\ddot{\mathbf{m}}(t-r/c)\times\hat{\mathbf{x}}\right]\times\hat{\mathbf{x}} + \frac{1}{6}\frac{\mu_0}{4\pi rc^2}\left[\dddot{\mathsf{Q}}(t-r/c)\hat{\mathbf{x}}\right]\times\hat{\mathbf{x}}\,. \qquad (6.28)$$

The electic field in the far-field is given by $\mathbf{E} = c\mathbf{B}\times\hat{\mathbf{x}}$.

The Poynting vector in the far-field is given by

$$\mathbf{N} = \frac{c}{\mu_0}|\mathbf{B}|^2\hat{\mathbf{x}}\,, \qquad (6.29)$$

as usual. If we integrate over all directions to determine the total radiated power, the cross-terms vanish between the various multipole contributions to the magnetic field leaving a sum of squares from the individual terms (exercise!):

$$\int \mathbf{N}\cdot d\mathbf{S} = \frac{\mu_0}{6\pi c}|\ddot{\mathbf{p}}|^2 + \frac{\mu_0}{6\pi c^3}|\ddot{\mathbf{m}}|^2 + \frac{\mu_0}{720\pi c^3}\dddot{Q}_{ij}\dddot{Q}_{ij}\,. \qquad (6.30)$$

The power from the magnetic dipole and electric quadrupole are generally suppressed by $(a/\lambda)^2 \sim (v/c)^2$ compared to the electric dipole, unless the latter vanishes for some reason.

7 Scattering

As an application of the material developed in the previous section on radiation of electromagnetic waves, we briefly consider the issue of scattering. An incident electromagnetic wave will cause a free charge, such as an electron, to oscillate and the acceleration that accompanies the oscillation will cause the charge to radiate. In this manner, the incident radiation is *scattered* into other directions. A similar thing happens for radiation incident on neutral systems such as atoms or molecules. As we shall discuss further in Part III, there are now bound positive and negative charges that displace from their equilibrium positions in response to the applied field. These displacements generate an oscillating electric dipole moment that radiates, scattering the incident radiation.

7.1 Thomson scattering

Thomson scattering describes the classical scattering of radiation from a free electron.[7] We shall only consider incident fields that are weak enough not to induce relativistic motions. In the frame in which the electron is at rest (on average), the equation of motion of the electron in the incident electromagnetic wave is simply

$$m_e\ddot{\mathbf{x}} = -e\mathbf{E}\,, \tag{7.1}$$

where m_e is the mass of the electron, $-e$ is the charge, and $\mathbf{E}$ is the incident field evaluated at the location of the electron. For an incident wave at angular frequency ω, we can evaluate the right-hand side of Eq. (7.1) at the average position of the electron provided that the wavelength of the incident wave is large compared to the amplitude of the oscillation induced upon the electron. In this case, the electron oscillates at frequency ω and with amplitude $eE_0/(m_e\omega^2)$, where E_0 is the amplitude of the electric field in the incident wave. This oscillation amplitude will be small compared to the wavelength provided that

$$\frac{eE_0}{m_e c\omega} \ll 1\,. \tag{7.2}$$

However, since the maximum speed of the electron is the product of the frequency and amplitude, i.e., $eE_0/(m_e\omega)$, Eq. (7.2) is equivalent to the requirement that the electron motion is non-relativistic and so will hold by assumption.

The acceleration of the electron is therefore $-e\mathbf{E}/m_e$ corresponding to an oscillating electric-dipole moment with

$$\ddot{\mathbf{p}} = e^2\mathbf{E}/m_e\,. \tag{7.3}$$

It follows that the time-averaged radiation power is

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{\mu_0}{6\pi c}\langle|\ddot{\mathbf{p}}|^2\rangle = \frac{\mu_0}{6\pi c}\frac{e^4E_0^2}{2m_e^2}\,. \tag{7.4}$$

The time-averaged magnitude of the Poynting vector of the incident radiation is

$$\langle|\mathbf{N}|\rangle = \frac{E_0^2}{2\mu_0 c}\,, \tag{7.5}$$

and the ratio of the radiated (scattered) power to this incident flux defines the *Thomson cross-section*,

$$\sigma_{\mathrm{T}} = \frac{\mu_0^2e^4}{6\pi m_e^2} = \frac{e^4}{6\pi\epsilon_0^2m_e^2c^4}\,. \tag{7.6}$$

[7] A quantum description is necessary when the energy of an incident photon is comparable to the rest mass energy of the electron. In this limit, the scattered radiation undergoes a frequency shift due to recoil of the electron.

This is the effective geometric cross-section of the electron to scattering. Note that it is independent of frequency. The numerical value of the Thomson cross-section is $\sigma_{\mathrm{T}} = 6.65 \times 10^{-29}\,\mathrm{m}^2$.

The Thomson cross-section can be neatly written in the form

$$\sigma_{\mathrm{T}} = \frac{8\pi}{3} r_e^2\,, \tag{7.7}$$

where the *classical electron radius* r_e is defined by

$$\frac{e^2}{4\pi\epsilon_0 r_e} = m_e c^2\,. \tag{7.8}$$

(One can think of r_e as being proportional to the radius that a uniform, extended charge e would have if the rest-mass energy were all electrostatic.)

7.2 Rayleigh scattering

The other type of scattering we shall consider here is scattering of long wavelength radiation ($\lambda \gg$ physical size of scatterer) from charges bound within systems that have no net charge. This is known as *Rayleigh scattering.* We shall see in Part III that for bound charges, provided that the frequency of the incident wave is small compared to any resonant frequency (or transition frequency in a quantum description), the bound charges will collectively displace following the applied force from the incident wave. This induces an oscillating electric dipole moment of the form $\mathbf{p} = \alpha\mathbf{E}$, where $\mathbf{E}$ is the incident electric field at the position of the (small) scatterer, and α is the *polarizability.*

The wavelength of the scattered radiation is the same as that of the incident radiation and is large compared to the size of the scatterer. We are therefore in the dipole limit, and the radiated power is given by Eq. (6.18). Since $\ddot{\mathbf{p}} = -\alpha\omega^2\mathbf{E}$, the time-averaged radiated power is

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{\mu_0}{6\pi c} \frac{\alpha^2\omega^4 E_0^2}{2}\,. \tag{7.9}$$

Dividing by the time-averaged incident energy flux gives the cross-section for Rayleigh scattering:

$$\sigma = \frac{\mu_0^2\alpha^2\omega^4}{6\pi} \propto \frac{1}{\lambda^4}\,. \tag{7.10}$$

Note the $1/\lambda^4$ wavelength dependence of the cross-section, so that short wavelengths (high frequencies) are scattered more than longer wavelengths. Rayleigh scattering underlies many natural phenomena. For example, the blue hue of the overhead sky is due to the (mostly) nitrogen and oxygen molecules in the atmosphere preferentially scattering the blue (short wavelength) component of sunlight. Furthermore, red sunsets follow from preferentially scattering blue light out of the line of sight towards the setting sun.

8 Radiation from an arbitrarily moving point charge

We now consider radiation from a single point charge moving arbitrarily, i.e., we relax the assumption of non-relativistic motion. Specifically, we consider a point charge q following a wordline $y^\mu(\tau)$ in spacetime, where τ is proper time. For such general motion, the dipole approximation may not be valid and its extension in terms of higher multipole moments will converge only very slowly. Here, we develop an alternative methodology to deal with this problem.

8.1 Lienard–Weichert potentials and fields

Recall from Eq. (3.52) that the current 4-vector for a point particle can be written covariantly as

$$J^\mu(x) = qc \int \delta^{(4)}\left(x - y(\tau)\right) \dot{y}^\mu(\tau)\, d\tau\,. \tag{8.1}$$

(Here, we are using the proper time τ to parameterise the path, rather than some general λ.) Inserting this into Eq. (5.15) for the retarded potential, we find

$$\begin{aligned} A^\mu(x) &= \frac{\mu_0 qc}{4\pi} \int d\tau \int d^4z\, \delta^{(4)}\left(z - y(\tau)\right) \dot{y}^\mu(\tau) \frac{\delta(x^0 - z^0 - |\mathbf{x} - \mathbf{z}|)}{|\mathbf{x} - \mathbf{z}|} \\ &= \frac{\mu_0 qc}{4\pi} \int \dot{y}^\mu(\tau) \frac{\delta\left(x^0 - y^0(\tau) - |\mathbf{x} - \mathbf{y}(\tau)|\right)}{|\mathbf{x} - \mathbf{y}(\tau)|} d\tau\,, \end{aligned} \tag{8.2}$$

where we performed the integration d^4z using the defining property of the delta-function. To make further progress, it is convenient to express the remaining 1D delta-function in covariant form using Eq. (5.19):

$$\frac{\delta\left(x^0 - y^0(\tau) - |\mathbf{x} - \mathbf{y}(\tau)|\right)}{|\mathbf{x} - \mathbf{y}(\tau)|} = 2\Theta\left(R^0(\tau)\right) \delta\left(\eta_{\mu\nu} R^\mu(\tau) R^\nu(\tau)\right)\,, \tag{8.3}$$

where we defined $R^\mu(\tau) = x^\mu - y^\mu(\tau)$. The combination of the delta-function and Heaviside function selects the unique value of proper time when the charge's worldline intersects the past lightcone of x^μ. Denoting the proper time there by τ_* (which is a function of x^μ), and using

$$\frac{d}{d\tau}\left[\eta_{\mu\nu} R^\mu(\tau) R^\nu(\tau)\right] = -2\eta_{\mu\nu} R^\mu(\tau) \dot{y}^\nu(\tau)\,, \tag{8.4}$$

we have

$$\frac{\delta\left(x^0 - y^0(\tau) - |\mathbf{x} - \mathbf{y}(\tau)|\right)}{|\mathbf{x} - \mathbf{y}(\tau)|} = \frac{\delta(\tau - \tau_*)}{|R^\mu(\tau_*) \dot{y}_\mu(\tau_*)|}\,. \tag{8.5}$$

Finally, using this in Eq. (8.2), and integrating over τ, we obtain the *Leinard–Wiechert potential* in covariant form:

$$A^{\mu}(x) = \frac{\mu_0 qc}{4\pi} \frac{\dot{y}^{\mu}(\tau_*)}{|R^{\nu}(\tau_*)\dot{y}_{\nu}(\tau_*)|} \,. \tag{8.6}$$

Equation (8.6) is a beautifully compact expression for the 4-potential due to an arbitrarily moving charge. However, to gain intuition for this result, it is useful to unpack it into time and space components in some inertial frame. Then, $\dot{y}^{\mu}(\tau_*) = \gamma(\tau_*)\,(c, \mathbf{v}(\tau_*))$, where $\mathbf{v}(\tau_*)$ is the 3-velocity at proper time τ_* and $\gamma(\tau_*)$ is the associated Lorentz factor. Furthermore, $R^{\mu}(\tau_*) = (c\Delta t(\tau_*), \mathbf{R}(\tau_*))$, where $\Delta t(\tau_*)$ is the time difference between the current event and the time that the charge intersected the past lightcone, and $\mathbf{R}(\tau_*) \equiv \mathbf{x} - \mathbf{y}(\tau_*)$. Note that $c\Delta t(\tau_*) = |\mathbf{R}(\tau_*)| = R(\tau_*)$. We now have

$$|R^{\nu}(\tau_*)\dot{y}_{\nu}(\tau_*)| = c\gamma(\tau_*)R(\tau_*)\left(1 - \hat{\mathbf{R}}(\tau_*)\cdot\mathbf{v}(\tau_*)/c\right) \,, \tag{8.7}$$

and the time and space components of Eq. (8.6) give

$$\phi(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0 R(\tau_*)\left(1 - \hat{\mathbf{R}}(\tau_*)\cdot\mathbf{v}(\tau_*)/c\right)} \tag{8.8}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0 q \mathbf{v}(\tau_*)}{4\pi R(\tau_*)\left(1 - \hat{\mathbf{R}}(\tau_*)\cdot\mathbf{v}(\tau_*)/c\right)} \,. \tag{8.9}$$

Due to the presence of the $1 - \hat{\mathbf{R}}(\tau_*)\cdot\mathbf{v}(\tau_*)/c$ factors in the denominators of Eqs. (8.8) and (8.9), the potentials (and hence fields) are concentrated in the direction of motion, $\hat{\mathbf{R}}(\tau_*) \propto \mathbf{v}(\tau_*)$, for relativistic motion. This is a consequence of extreme aberration for a rapidly moving particle.

If we adopt an inertial frame such that the particle is at rest in that frame at τ_*, the Lienard–Wiechert potentials simplify to $\mathbf{A}(t, \mathbf{x}) = 0$ and

$$\phi(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0 R_{\text{rest}}(\tau_*)} \qquad \text{(rest-frame)} \,, \tag{8.10}$$

where $R_{\text{rest}}(\tau_*)$ is the distance between the observation event and $y^{\mu}(\tau_*)$ in this rest-frame. These *potentials* are exactly what one would expect for a stationary point charge. If instead we adopt a frame in which the charge is not at rest at τ_*, but is moving non-relativistically, the vector potential no longer vanishes but is approximately

$$\mathbf{A}(t, \mathbf{x}) \approx \frac{\mu_0 q \mathbf{v}(\tau_*)}{4\pi R(\tau_*)} \qquad \text{(non-relativistic)} \,. \tag{8.11}$$

This is consistent with the non-relativistic (dipole) approximation in Eq. (6.7) noting that $d\mathbf{p}/dt = q\mathbf{v}$ for a point charge (where $\mathbf{p}$ is the electric dipole moment, not the 3-momentum!).

Example: The fields of a charge in uniform motion. Consider a charge q moving with velocity $\mathbf{v}$ such that its position at time t is $\mathbf{y}(t) = \mathbf{v}t$. The potentials at $(ct, \mathbf{x})$ are generated at the earlier time t_*, such that the spatial separation $\mathbf{R} = \mathbf{x} - \mathbf{v}t_*$ and temporal separation $\Delta t \equiv t - t_*$ are the component of a null vector:

$$c\Delta t = |\mathbf{x} - \mathbf{v}t_*| \,. \tag{8.12}$$

For the uniform motion considered in this example, $\mathbf{R}$ (and so Δt) depends only on the displacement from the charge's *current* position,

$$\mathbf{r} \equiv \mathbf{x} - \mathbf{v}t \,. \tag{8.13}$$

To see this, we note that the null condition becomes

$$c\Delta t = |\mathbf{r} + \mathbf{v}\Delta t| \,, \tag{8.14}$$

which determines Δt in terms of $\mathbf{r}$, and

$$\mathbf{R} = \mathbf{r} + \mathbf{v}\Delta t \,. \tag{8.15}$$

We solve for Δt by squaring both sides of Eq. (8.14) to find

$$c\Delta t/\gamma^2 = \mathbf{r} \cdot \mathbf{v}/c + \sqrt{(\mathbf{r} \cdot \mathbf{v})^2/c^2 + r^2/\gamma^2} \,, \tag{8.16}$$

where we have used $\Delta t > 0$ to select the appropriate root. The denominators in Eqs. (8.8) and (8.9) involve the combination $R - \mathbf{R} \cdot \mathbf{v}/c$. Noting that $R = c\Delta t$, we have

$$\begin{aligned} R - \mathbf{R} \cdot \mathbf{v}/c &= R - \mathbf{r} \cdot \mathbf{v}/c - v^2 \Delta t/c \\ &= c\Delta t/\gamma^2 - \mathbf{r} \cdot \mathbf{v}/c \,, \end{aligned} \tag{8.17}$$

so that

$$R - \mathbf{R} \cdot \mathbf{v}/c = \sqrt{(\mathbf{r} \cdot \mathbf{v})^2/c^2 + r^2/\gamma^2} \,. \tag{8.18}$$

This allows us to write the potentials as

$$\phi(t, \mathbf{x}) = \frac{\gamma q}{4\pi\epsilon_0 \sqrt{r^2 + \gamma^2 (\mathbf{r} \cdot \mathbf{v})^2/c^2}} \tag{8.19}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0 \gamma q \mathbf{v}}{4\pi \sqrt{r^2 + \gamma^2 (\mathbf{r} \cdot \mathbf{v})^2/c^2}} \,. \tag{8.20}$$

We can compute the fields from $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and $\mathbf{E} = -\partial \mathbf{A}/\partial t - \boldsymbol{\nabla}\phi$, recalling that $\mathbf{r} = \mathbf{x} - \mathbf{v}t$. For the electric field, this gives

$$\mathbf{E} = \frac{q\gamma \mathbf{r}}{4\pi\epsilon_0 \left[\gamma^2 (\mathbf{r} \cdot \mathbf{v})^2/c^2 + \mathbf{r}^2\right]^{3/2}} \,, \tag{8.21}$$

and the magnetic field is $\mathbf{B} = \mathbf{v} \times \mathbf{E}/c^2$. We see that the electric field is radially out from the *current* position of the charge, and the magnetic field is azimuthal, and both are anisotropic.

For $\mathbf{r}$ along the direction of motion, the field is smaller by a factor of $1/\gamma^2$ than for a charge at rest at the current position, while perpendicular to this the field is enhanced by a factor of γ over that for a charge at rest. The field lines are thus concentrated close to the plane perpendicular to the motion for a rapidly moving particle. Note that this does not contradict our earlier finding that the fields are concentrated where $\hat{\mathbf{R}}$ is along $\mathbf{v}$. If we ask how the fields vary around a circle of radius r centred on the current position of the charge, we also have to take account that the fields were "generated" at times t_* that have a strong dependence on the direction $\hat{\mathbf{r}}$. In particular, for $\hat{\mathbf{r}}$ along the direction of motion, $c\Delta t$ becomes very large for a rapidly moving particle (going as $2r\gamma^2$ for $v \approx c$), since the particle almost keeps up with light signals emitted in the forward direction, and so the fields were generated when the particle was very far away. This overcomes the intrinsic field concentration along $\hat{\mathbf{R}} = \hat{\mathbf{v}}$.

Finally, we note that the fields due to a uniformly-moving charge can also be computed from those for a static charge by Lorentz transformation (see IB *Electromagnetism*).

To calculate the electric and magnetic fields of an arbitrarily moving charge, we return to the covariant form of the Lienard–Weichert potentials (Eq. 8.6) and form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We have to remember that τ_* is a function of x^μ, so that, for example, $\partial_\mu \dot{y}^\nu(\tau_*) = \ddot{y}^\nu(\tau_*)\partial_\mu\tau_*$. We can evaluate $\partial_\mu\tau_*$ by differentiating the null condition $\eta_{\alpha\beta}R^\alpha(\tau_*)R^\beta(\tau_*) = 0$ to find

$$\begin{aligned} 0 &= \eta_{\alpha\beta}\partial_\mu R^\alpha(\tau_*)R^\beta(\tau_*) \\ &= \eta_{\alpha\beta}\partial_\mu\left(x^\alpha - y^\alpha(\tau_*)\right)R^\beta(\tau_*) \\ &= \eta_{\alpha\beta}\left(\delta^\alpha_\mu - \dot{y}^\alpha(\tau_*)\partial_\mu\tau_*\right)R^\beta(\tau_*)\,. \end{aligned} \tag{8.22}$$

Solving for $\partial_\mu\tau_*$ gives

$$\frac{\partial\tau_*}{\partial x^\mu} = \frac{R_\mu(\tau_*)}{R^\nu(\tau_*)\dot{y}_\nu(\tau_*)}\,. \tag{8.23}$$

The other term we require in differentiating Eq. (8.6) is

$$\begin{aligned} \partial_\mu\left[R^\nu(\tau_*)\dot{y}_\nu(\tau_*)\right] &= \left(\delta^\nu_\mu - \dot{y}^\nu(\tau_*)\partial_\mu\tau_*\right)\dot{y}_\nu(\tau_*) + R^\nu(\tau_*)\ddot{y}_\nu(\tau_*)\partial_\mu\tau_* \\ &= \partial_\mu\tau_*\left[R^\nu(\tau_*)\ddot{y}_\nu(\tau_*) - \dot{y}^\nu(\tau_*)\dot{y}_\nu(\tau_*)\right] + \dot{y}_\mu(\tau_*) \\ &= \partial_\mu\tau_*\left[c^2 + R^\nu(\tau_*)\ddot{y}_\nu(\tau_*)\right] + \dot{y}_\mu(\tau_*)\,. \end{aligned} \tag{8.24}$$

Putting these pieces together, we find (exercise!), on antisymmetrising,

$$F_{\mu\nu}(x) = -\frac{\mu_0 qc}{4\pi(cR_{\text{rest}})}\left(\frac{R_\mu\ddot{y}_\nu - R_\nu\ddot{y}_\mu}{(cR_{\text{rest}})} + \frac{(R_\mu\dot{y}_\nu - R_\nu\dot{y}_\mu)}{(cR_{\text{rest}})^2}(c^2 + R^\alpha\ddot{y}_\alpha)\right)\,, \tag{8.25}$$

where all quantities on the right are evaluated at τ_*. Here, we have used

$$-R^\mu(\tau_*)\dot{y}_\mu(\tau_*) = cR_{\text{rest}}\,, \tag{8.26}$$

to express $F_{\mu\nu}$ in terms of $R_{\rm rest}$, the radial distance between x^μ and $y^\mu(\tau_*)$ as measured in the frame in which the charge is at rest at τ_*.

Inspection of Eq. (8.26) reveals terms going like $1/R$, that depend on the 4-acceleration, and a term going like $1/R^2$ that depends only on the 4-velocity of the charge. The $1/R$ terms describe radiation, while the $1/R^2$ term turns out to be the field that would be generated from a charge moving uniformly with the same velocity at τ_* (see below).

Exercise: Extract the electric and magnetic fields from the field-strength tensor in Eq. (8.26) in a general inertial frame, recalling that the 4-acceleration has components

$$\ddot{y}^\mu = \gamma^2 \left(\frac{\gamma^2}{c} \mathbf{v} \cdot \mathbf{a}, \frac{\gamma^2}{c^2} \mathbf{a} \cdot \mathbf{v} \mathbf{v} + \mathbf{a} \right) . \tag{8.27}$$

You should find (after a lengthy, but straightforward, calculation) that

$$\mathbf{E}(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0 (1 - \mathbf{v} \cdot \hat{\mathbf{R}}/c)^3} \left(\frac{\hat{\mathbf{R}} - \mathbf{v}/c}{\gamma^2 R^2} + \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \mathbf{v}/c) \times \mathbf{a}]}{R c^2} \right) , \tag{8.28}$$

where all quantities on the right are evaluated at τ_*. For the magnetic field, you should find

$$\mathbf{B}(t, \mathbf{x}) = \frac{1}{c} \hat{\mathbf{R}}(\tau_*) \times \mathbf{E}(t, \mathbf{x}) . \tag{8.29}$$

The exercise above derives the electric and magnetic fields from $F_{\mu\nu}$. Note the following points.

- The magnetic field is always perpendicular to the electric field and to $\hat{\mathbf{R}}(\tau_*)$.
- The radiation ($1/R$) part of the fields is sourced by the 3-acceleration. The electric field has an angular dependence given by $\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \mathbf{v}/c) \times \mathbf{a}]$, and is therefore perpendicular to $\hat{\mathbf{R}}(\tau_*)$, like the magnetic field. The angular dependence is more complicated than for dipole radiation [Eq. (6.14)], due to the presence of the $\mathbf{v}/c$ correction to $\hat{\mathbf{R}}$. However, in the non-relativistic limit ($|\mathbf{v}| \ll c$), the radiation fields reduce to those derived in the dipole approximation in Sec. 6 (noting that $d^2\mathbf{p}/dt^2 = q\mathbf{a}$ for a point charge). The magnitudes of the fields satisfy the usual relation for radiation, $|\mathbf{E}| = c|\mathbf{B}|$, even for relativistic motion of the charge.
- The $1/R^2$ contribution to the fields are the same as for a charge moving uniformly with the same velocity at τ_*. To see this, recall the following expression for the

electric field from a uniformly moving charge (given above in Eq. 8.21):

$$\mathbf{E} = \frac{q\gamma\mathbf{r}}{4\pi\epsilon_0 \left[\gamma^2(\mathbf{r}\cdot\mathbf{v})^2/c^2 + \mathbf{r}^2\right]^{3/2}}\,, \tag{8.30}$$

where $\mathbf{r}$ points from the *current* position of the charge to the observation point. We can express this back in terms of $\mathbf{R}(\tau_*)$ using Eq. (8.18) and

$$\mathbf{r} = \mathbf{R} - \mathbf{v}\Delta t = R(\hat{\mathbf{R}} - \mathbf{v}/c)\,, \tag{8.31}$$

which recovers exactly the first $(1/R^2)$ term in Eq. (8.28).

8.2 Power radiated

We now calculate the rate at which energy is radiated by an arbitrarily moving point charge. We shall first calculate this in the instantaneous rest frame of the charge and then use Lorentz transformations to deduce the rate in a general inertial frame.

In the frame in which the particle is at rest at proper time τ_*, the radiation part of electric field on the forward lightcone through $y^\mu(\tau_*)$ is, from Eq. (8.28),

$$E(t,\mathbf{x}) = \frac{q}{4\pi\epsilon_0 R_{\text{rest}} c^2}\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \mathbf{a}_{\text{rest}}) \qquad \text{(Instantaneous rest frame)}\,, \tag{8.32}$$

where $\mathbf{a}_{\text{rest}}$ is the 3-acceleration in the instantaneous rest frame. The magnetic field is $\mathbf{B} = \hat{\mathbf{R}} \times \mathbf{E}/c$, so the Poynting vector is

$$\begin{aligned}\mathbf{N} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \\ &= \frac{1}{c\mu_0}\mathbf{E}\times(\hat{\mathbf{R}}\times\mathbf{E}) \\ &= \frac{1}{c\mu_0}|\mathbf{E}|^2\hat{\mathbf{R}}\,,\end{aligned} \tag{8.33}$$

which is radial. We can get the radiated power by integrating over a spherical surface of radius R_{rest} at time t (i.e., a section through the forward lightcone corresponding to a constant time surface in the instantaneous rest frame). Since the charge is at rest at τ_*, the radiation emitted in proper time $d\tau$ around τ_* crosses this spherical surface in (coordinate) time[8] $dt = d\tau$. It follows that the energy radiated in the instantaneous

[8] This would not be true if the charge were moving at τ_*. Generally, the time taken for the radiation emitted in $d\tau_*$ to cross a spherical surface of radius $R(\tau_*)$ can be got from Eq. (8.23). Rearranging gives

$$d\tau_* = \frac{R_\mu(\tau_*)dx^\mu}{R^\nu(\tau_*)\dot{y}_\nu(\tau_*)}\,.$$

rest frame in proper time $d\tau$ is

$$
\begin{aligned}
dE_{\rm rest} &= d\tau \int \mathbf{N}\cdot d\mathbf{S} \\
&= \frac{R_{\rm rest}^2 d\tau}{c\mu_0}\int |\mathbf{E}|^2\, d\Omega\,,
\end{aligned}
\tag{8.34}
$$

where $d\Omega$ is the element of solid angle (i.e., $d\Omega = d^2\hat{\mathbf{R}}$). Taking the rest-frame acceleration to be along the z-axis, we have

$$
|\hat{\mathbf{R}}\times(\hat{\mathbf{R}}\times\mathbf{a}_{\rm rest})| = \sin\theta|\mathbf{a}_{\rm rest}|\,,
\tag{8.35}
$$

so that on integrating over solid angle

$$
\frac{dE_{\rm rest}}{d\tau} = \frac{\mu_0 q^2|\mathbf{a}_{\rm rest}|^2}{6\pi c}\,.
\tag{8.36}
$$

This simple result is exactly what we would have got by applying the dipole approximation, Eq. (6.18), in the instantaneous rest frame, with $d^2\mathbf{p}/dt^2 \to q\mathbf{a}_{\rm rest}$. This is not unexpected – the dipole approximation holds in situations where all velocities are non-relativistic and this is exactly true in the instantaneous rest frame. Indeed, the dipole approximation provides a useful short-cut to deriving Eq. (8.36).

Now consider the 4-momentum radiated in time $d\tau$. Since equal power is emitted along $\hat{\mathbf{R}}$ and $-\hat{\mathbf{R}}$ in the instantaneous rest frame, the total 3-momentum of the radiation is zero. We therefore have

$$
dP^\mu = d\tau\left(\frac{1}{c}\frac{dE_{\rm rest}}{d\tau},\mathbf{0}\right)
\tag{8.37}
$$

in the rest frame. Lorentz transforming to an inertial frame in which the charge has velocity $\mathbf{v}(\tau_*)$ at τ_*, we have

$$
\frac{dE'}{c} = dP'^0 = \gamma(\tau_*)dP^0 = \frac{\gamma(\tau_*)d\tau}{c}\frac{dE_{\rm rest}}{d\tau}\,.
\tag{8.38}
$$

The time elapsed between the events $y^\mu(\tau_*)$ and $y^\mu(\tau_* + d\tau)$ in the general frame is $dt' = \gamma(\tau_*)d\tau$ by time dilation, hence

$$
\frac{dE'}{dt'} = \frac{dE_{\rm rest}}{d\tau}\,.
\tag{8.39}
$$

Taking $dx^\mu = (cdt,\mathbf{0})$ gives

$$
d\tau_* = \frac{dt}{\gamma(\tau_*)\left[1-\hat{\mathbf{R}}(\tau_*)\cdot\mathbf{v}(\tau_*)/c\right]}\,.
$$

This is just the usual Doppler effect. In the instantaneous rest frame, it reduces to $d\tau_* = dt$, as advertised.

We see that the rate at which energy is radiated by the accelerated charge is Lorentz invariant and so, generally, the instantaneous power radiated is

$$\frac{dE}{dt} = \frac{\mu_0 q^2 a_\mu a^\mu}{6\pi c} = \frac{\mu_0 q^2 \gamma^4}{6\pi c}\left[\mathbf{a}^2 + \gamma^2(\mathbf{a}\cdot\mathbf{v}/c)^2\right] , \tag{8.40}$$

where a^μ is the acceleration 4-vector (so that $a^\mu a_\mu = |\mathbf{a}_{\rm rest}|^2$). This result is known as the *relativistic Larmor formula.* (Its non-relativistic limit was first worked out by Larmor in 1897.)

Non-examinable aside: If the above argument is not to your taste, the same expression for the power radiated in a general frame can be obtained by direct calculation. The radiation part of the electric field is now given by

$$\mathbf{E}(t,\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\mathbf{v}/c)\times\mathbf{a}]}{R(1-\mathbf{v}\cdot\hat{\mathbf{R}}/c)^3c^2} , \tag{8.41}$$

where all quantities on the right are evaluated at τ_*. The Poynting vector is still given by Eq. (8.33), but when integrating this over the surface of a sphere of radius $R(\tau_*)$ at time t, to determine the radiated power, we have to be careful to account for Doppler shifts that relate *dt at the source* to the increment in time for radiation to pass through the radius $R(\tau_*)$. Following the arguments in the footnote on Page 49, and noting that the coordinate time between the events at τ_* and $\tau_* + d\tau$ on the charge's worldline is $\gamma(\tau_*)d\tau$, the power emitted per solid angle (i.e., energy per solid angle per coordinate time *at source*) is

$$\begin{aligned}\frac{dE}{dtd\Omega} &= \frac{1}{c\mu_0}\left(1-\frac{\mathbf{v}\cdot\hat{\mathbf{R}}}{c}\right)|\mathbf{E}|^2R^2 \\ &= \frac{\mu_0 q^2}{(4\pi)^2c}\frac{\left|\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\mathbf{v}/c)\times\mathbf{a}]\right|^2}{\left(1-\mathbf{v}\cdot\hat{\mathbf{R}}/c\right)^5} .\end{aligned} \tag{8.42}$$

Note that the angular dependence of the radiated power is more complicated than the $\sin^2\theta$ dependence in the instantaneous rest frame. In particular, for a particle moving relativistically, the power is significantly concentrated in the direction of motion, $\hat{\mathbf{R}} \propto \mathbf{v}$. After some vector algebra, Eq. (8.42) can be reduced to

$$\frac{dE}{dtd\Omega} = \frac{\mu_0 q^2}{(4\pi)^2 c}\left(2\frac{(\hat{\mathbf{R}}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{v}/c)}{\left(1-\mathbf{v}\cdot\hat{\mathbf{R}}/c\right)^4} + \frac{|\mathbf{a}|^2}{\left(1-\mathbf{v}\cdot\hat{\mathbf{R}}/c\right)^3} - \frac{(\hat{\mathbf{R}}\cdot\mathbf{a})^2}{\gamma^2\left(1-\mathbf{v}\cdot\hat{\mathbf{R}}/c\right)^5}\right) . \tag{8.43}$$

Integrating over solid angle[9] directly gives the final result in Eq. (8.40).

---

[9]A straightforward, but lengthy, way to do this is to align $\mathbf{v}$ with the z-axis and $\mathbf{a}$ in the x-z plane, and express the components of $\hat{\mathbf{R}}$ in spherical polar coordinates.

8.3 Synchrotron radiation

We end our discussion of the radiation from arbitrarily moving charged particles with a specific example. Synchrotron radiation refers to the radiation emitted by a charged particle moving relativistically in circular motion (typically under the action of a magnetic field).

Consider a particle of charge q and mass m moving in a circle with speed v perpendicular to a magnetic field of magnitude B. We saw in Sec. 3.1.1 that such a particle has angular velocity $\omega = qB/(\gamma m)$ and so the acceleration has magnitude

$$|\mathbf{a}| = \frac{qBv}{\gamma m}\,, \tag{8.44}$$

and is perpendicular to $\mathbf{v}$. The total instantaneous power radiated follows from Eq. (8.40), which here gives

$$\frac{dE}{dt} = \frac{\mu_0}{6\pi c}\left(\frac{q^2 B\gamma v}{m}\right)^2 . \tag{8.45}$$

Note that this is proportional to the square of the 3-momentum of the particle.

As discussed earlier, for a relativistic particle the emitted power is concentrated (or *beamed*) into a narrow range of solid angles around the instantaneous velocity. The beaming is controlled by factors of $1-\mathbf{v}\cdot\hat{\mathbf{R}}/c$ in the denominator of the emitted power (see e.g., Eq. 8.42). We can estimate the angular extent of the beamed radiation by asking for what angle θ between $\mathbf{v}$ and $\hat{\mathbf{R}}$ does $(1-\mathbf{v}\cdot\hat{\mathbf{R}}/c)^{-1}$ fall to one-half its peak value, i.e.,

$$1-\beta\cos\theta = 2(1-\beta)\,, \tag{8.46}$$

where $\beta = v/c$. Writing $\beta = 1-\epsilon$, and working to leading order in ϵ, we find $\theta^2 = 2\epsilon$. We can write this in terms of the Lorentz factor γ noting that

$$\gamma^{-2} = 1-\beta^2 = 2\epsilon + \cdots\,, \tag{8.47}$$

so that $\theta \approx 1/\gamma$. We see that for a relativistic particle with $\gamma \gg 1$, the emitted radiation is beamed into a cone in the direction of motion with an opening angle of the order of $1/\gamma$.

Now consider observing the synchotron radiation at position $\mathbf{x}$ with the particle's orbit centred on the origin. For $|\mathbf{x}|$ large compared to the radius of the orbit, we are safely in the far-field and can also approximate $\hat{\mathbf{R}}$ with $\hat{\mathbf{x}}$. The argument above tells us that only observers in the plane of the orbit will observe significant radiation. Since the orbit is periodic with period ω, so is the observed radiation. However, due to the beaming the radiation is received in short pulses (think of observing a rotating searchlight). We can estimate the duration of the pulses by noting that each pulse is emitted with duration Δt_* at source, such that the particle moves through an angle equal to twice the opening angle of the beamed radiation pattern, i.e., $\omega \Delta t_* = 2/\gamma$. In relating this to the duration of the observed pulse we have to taken account of the Doppler effect. Generally, an increment in retarded (coordinate) time corresponds to an increment in time at the observation point with

$$dt = (1 - \hat{\mathbf{R}} \cdot \mathbf{v}/c) dt_* \tag{8.48}$$

(see the footnote on Page 49), so that for the duration of the pulse

$$\Delta t = (1 - \beta) \Delta t_* \approx \frac{1}{\gamma^3 \omega} \,. \tag{8.49}$$

This means that the radiation is observed to have Fourier components up to roughly $\gamma^3 \omega$, and so a relativistic particle emits a broad frequency spectrum of radiation up to γ^3 times the orbital frequency.

Part III

Electromagnetism in media

So far in this course we have considered isolated charges and currents in vacuum. However, for many practical applications we are interested in electric and magnetic fields inside media, including solids, liquids and gases. Here, the story is rather more complicated. We have to consider the distribution of charges within the atoms/molecules that make up the medium, and the currents generated by the motion of these charges. Moreover, we need to consider how these distributions change in the presence of externally-applied fields. Generally, the electromagnetic fields inside media are very complicated with significant variation on the atomic scale. However, we are often only interested in the averaged properties of fields in regions large compared to the atomic scale. For example, to consider the propagation of light of wavelength $\lambda \sim O(500)\,\text{nm}$ through some medium, we can effectively average the medium on scales small compared to λ but large compared to atomic scales, $O(0.1)\,\text{nm}$. This leads us to consider an effective or macroscopic version of the Maxwell equations that describe the fields averaged over macroscopically-small but microscopically-large regions.

In this final part of the course, we shall first briefly introduce some of the basic electromagnetic properties of matter (Sec. 9). We then develop the macroscopic versions of Maxwell's equations in Sec. 10 by a spatial averaging procedure. The rest of the course applies these macroscopic equations to wave propagation in dielectrics (e.g., materials like glass; Secs 11 and 12) and conductors (Sec. 13).

9 Electromagnetic properties of matter

We shall mostly be concerned with non-conducting media. There is still tremendous diversity in the electromagnetic properties of such materials, but we shall only be concerned with some of the simplest examples here.

9.1 Dielectric media

An important class of insulating materials are called *dielectrics*. These have no mobile charges that can move freely in an applied field (unlike a conductor) but they nevertheless have a significant effect on applied electric fields. Generally, the individual atoms/moleclues that make up the material may have complicated time-dependent charge distributions. For example, water molecules have strong electric dipole moments and these vary on a short timescale due to thermal motion and internal vibration of the molecules. When averaged over macroscopic volumes, which contain a great many molecules, these dipole moments tend to cancel out in the absence of an applied field. However, if a field is applied, there will be coherent perturbations to the charge distribution of molecules within macroscopic volumes that will survive averaging. This phenomena is known as *polarisation*. For example, materials with intrinsic but randomly-oriented molecular dipole moments will tend to align their moments due to the torque of the applied field, while materials with no initial dipole moments will generally develop aligned moments as the applied field distorts the equilibrium charge distribution in each molecule. It is these coherent effects that determine the macroscopic behaviour of electromagnetic fields in media.

Toy model for atomic polarisability: Quantum theory is needed to describe the polarisation of atoms in applied fields. However, the following simple classical toy model is useful to develop understanding and to give an order-of-magnitude estimate of the induced polarisation.

Consider a point particle (the nucleus) with positive charge q, surrounded by an electron cloud of total charge $-q$ and radius a. Assume the electron cloud to have uniform charge density. At radius r from the centre of the cloud, the charge enclosed is $-q(r/a)^3$, and so the electric field at position $\mathbf{x}$ relative to the centre is

$$\mathbf{E}_{\text{cloud}}(\mathbf{x}) = -\frac{q}{4\pi\epsilon_0 a^3}\mathbf{x} \,. \tag{9.1}$$

If we apply a uniform external field $\mathbf{E}$, the positive charge will displace from the centre of the cloud so that the total force on it is zero. (We ignore any distortion of the cloud.) If the displacement is $\mathbf{d}$, we must have

$$q\mathbf{d} = 4\pi\epsilon_0 a^3 \mathbf{E} \quad \Rightarrow \quad \mathbf{p} = 4\pi\epsilon_0 a^3 \mathbf{E} \,. \tag{9.2}$$

Note that the induced dipole moment $\mathbf{p}$ is proportional to $\mathbf{E}$. It will generally be the case that $\mathbf{p}$ is linear in $\mathbf{E}$ since this is the first term in a Taylor expansion of $\mathbf{p}(\mathbf{E})$ given that $\mathbf{p}(\mathbf{E}=0)=0$. Higher terms in the expansion are negligible if the applied field is much smaller than the typical field within the atom (around $q/(4\pi\epsilon_0 a^3)$ in this example) or, equivalently, the displacements are small compared to the atomic scale. If we define the *polarisability* α by

$$\mathbf{p} = \alpha \mathbf{E} \,, \tag{9.3}$$

we see that $\alpha = 4\pi\epsilon_0 a^3$. We can estimate α for hydrogen atoms by taking $a \sim 10^{-10}\,\text{m}$ (a typical atomic size from quantum theory) to find $\alpha/(4\pi\epsilon_0) \sim 10^{-30}\,\text{m}^3$; this is close to the measured value $\alpha/(4\pi\epsilon_0) = 6.67 \times 10^{-31}\,\text{m}^3$. The induced displacement d depends on the magnitude of the electric field. For a field of $10^5\,\text{V}\,\text{m}^{-1}$ (i.e., $100\,\text{V}$ over $1\,\text{mm}$), the displacement in hydrogen is $d = 5 \times 10^{-17}\,\text{m}$, much smaller than the size of the atom.

9.2 Magnetic media

Magnetic fields are generated by charges in motion, i.e., currents. The leading-order magnetic field far away from a localised current distribution can be expressed in terms of the *magnetic dipole moment*

$$\mathbf{m} = \frac{1}{2}\sum_i q_i \mathbf{x}_i \times \mathbf{v}_i \,, \tag{9.4}$$

where the sum is over the charges, with the ith charge at position $\mathbf{x}_i$ and having velocity $\mathbf{v}_i$. The continuum form of this is

$$\mathbf{m} = \frac{1}{2}\int (\mathbf{x} \times \mathbf{J})\, d^3\mathbf{x} \,, \tag{9.5}$$

where the current density $\mathbf{J}(t, \mathbf{x}) = \sum_i q_i \mathbf{v}_i(t)\delta^{(3)}(\mathbf{x} - \mathbf{x}_i(t))$. The orbiting electrons in an atom/molecule may give rise to a net magnetic moment. There can also be a contribution from the intrinsic *spin* of the electrons – a purely quantum mechanical effect. The magnetic response of materials to applied fields depends on whether the net magnetic moment of each constituent molecule vanishes or not in the absence of an applied magnetic field. The phenomenology is rather richer than the electrical response of dielectrics discussed above.

- *Diamagnetism.* A material is diamagnetic if the net magnetic moments vanish when there is no applied field. Once a field is applied, the orbital motions are slightly distorted to generate magnetic moments that tend to be anti-parallel to the applied field. One can think of this in terms of Lenz's law: the induced current flows in a direction to oppose the change in magnetic field. A simple (classical) toy model for diamagnetism is given below. Most metals are diamagnetic.

- *Paramagnetism.* A material is paramagnetic if the constituent molecules have non-zero intrinsic magnetic moments. In this case, the magnetic moments will tend to be randomly oriented when there is no applied magnetic field, giving magnetic fields inside the medium that vary rapidly in time and space. However, on averaging over a macroscopic volume, the fields average down. Once an external magnetic field is applied, the intrinsic magnetic moments tend to align along the field due to the torque acting on them, giving a net effect that survives volume averaging. The response is strongly temperature dependent (as is the electric response for polar dielectric materials), being weaker at high temperatures due to the randomising effect of thermal motions.

- *Ferromagnetism.* Ferromagnets are paramagnetic, but strong quantum-mechanical (exchange) interactions between the electrons in neighbouring constituents give radically different behaviour. At temperatures below what is called the *Curie* *temperature*, the interactions are strong enough to allow spontaneous magnetisation: the magnetic moments are aligned in macroscopically-large regions called domains, even in the absence of an applied field. Applying a field changes the domains, and the moments in different domains tend to align along the applied field. When the external field is removed, some net alignment across domains remains producing a strong (intrinsic) magnetic field. Ferromagnets are what we usually think of as "magnets". The most common example is iron.

Toy model for atomic diamagnetism: Quantum theory is needed to describe properly the magnetic response of the electrons in an atom to an applied magnetic field[10]. However, the following simple classical calculation captures the basic physics at work.

Consider a charge q of mass m orbiting about a nucleus at radius a. Let the orbit lie normal to the z-direction and the charge be moving anti-clockwise. In the absence of an applied magnetic field, the Coulomb force due to the nucleus, of magnitude F_C, provides the centripetal force such that

$$F_C = mv^2/a\,, \tag{9.6}$$

where v is the orbital speed. Now apply a magnetic field B along the positive z-axis. If we ignore any change in the shape and size of the orbit, the orbital speed will change to $v + \delta v$ such that

$$F_C - q(v+\delta v)B = m(v+\delta v)^2/a\,, \tag{9.7}$$

i.e., the change in the radial force due to the magnetic field equals the change in the centripetal force. Expanding to first order in δv and B gives

$$\delta v = -\frac{qa}{2m}B\,. \tag{9.8}$$

As a result, the magnetic moment of the orbiting charge is changed by an amount

$$\delta m = -\frac{q^2a^2}{4m}B \tag{9.9}$$

along the z-direction. Note that the change in magnetic field is *anti-parallel* to the applied field. If the particle were orbiting clockwise instead, the speed would *increase* by $qaB/(2m)$, but the change in the magnetic moment would still be given by Eq. (9.9).

Diamagnetism is a very weak effect. Taking q to be the charge of an electron, m the electron mass, $a \sim 10^{-10}\,\mathrm{m}$, and $B = 1\,\mathrm{T}$ (a strong magnetic field in the laboratory), we find $\delta m \sim 10^{-28}\,\mathrm{A\,m^2}$. This should be compared to the natural atomic unit of magnetic moment, the *Bohr magneton*:

$$\mu_\mathrm{B} \equiv \frac{e\hbar}{2m_e} = 9.27\times 10^{-24}\,\mathrm{A\,m^2}\,. \tag{9.10}$$

[10]Indeed, the Bohr–van Leeuwen theorem forbids any non-zero thermal average of the magnetic moment for a *classical* system in thermal equilibrium.

(The Bohr magneton is approximately the magnetic moment due to the spin of an electron, and is also the z-component of the magnetic moment due to orbital motion of an electron with angular momentum $L_z = \hbar$.) We see that, even for a strong field of 1 T, the induced magnetic moment is only $O(10^{-5})\mu_{\mathrm{B}}$.

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10 Macroscopic Maxwell equations

10.1 Averaging the microscopic Maxwell equations

In classical physics, on scales large compared to nuclear dimensions, $O(10^{-15}\,\mathrm{m})$, we can treat atoms as collections of point particles. The *microscopic* Maxwell equations inside media are the usual ones, with the charges and currents due to a collection of (moving) point charges. We denote the rapidly-varying electric and magnetic fields inside the medium by $\mathbf{E}_{\mathrm{micro}}$ and $\mathbf{B}_{\mathrm{micro}}$, so that

$$\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathrm{micro}} = \frac{\rho_{\mathrm{micro}}}{\epsilon_0} \tag{10.1}$$

$$\boldsymbol{\nabla} \times \mathbf{E}_{\mathrm{micro}} = -\frac{\partial \mathbf{B}_{\mathrm{micro}}}{\partial t} \tag{10.2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B}_{\mathrm{micro}} = 0 \tag{10.3}$$

$$\boldsymbol{\nabla} \times \mathbf{B}_{\mathrm{micro}} = \mu_0 \mathbf{J}_{\mathrm{micro}} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}_{\mathrm{micro}}}{\partial t}\,, \tag{10.4}$$

with ρ_{micro} and $\mathbf{J}_{\mathrm{micro}}$ the microscopic charge and current densities, respectively.

If we are only interested in lengthscales large compared to atomic dimensions, we can average the microscopic fields to obtain smooth *macroscopic* fields that we denote by $\mathbf{E}$ and $\mathbf{B}$.[11] The averaging is done by convolution with some smoothing function $f(\mathbf{x})$, so that, for example,

$$\mathbf{E}(t, \mathbf{x}) \equiv \langle \mathbf{E}_{\mathrm{micro}}(t, \mathbf{x}) \rangle = \int f(\mathbf{x}') \mathbf{E}_{\mathrm{micro}}(t, \mathbf{x} - \mathbf{x}')\, d^3\mathbf{x}'\,. \tag{10.5}$$

We do not have to be precise about the choice of smoothing function, but it should be real, smoothly localised to some macroscopic region about $\mathbf{x} = 0$, and integrate to unity over all space. It is usual to take $f(\mathbf{x})$ to be spherically symmetric to preserve any directional characteristics of the medium. The averaging operation commutes with

[11] In the earlier parts of this course we were concerned with isolated charges in vacuum. In this situation, there is no distinction between the macroscopic and microscopic fields well away from any charges.

differentiation in time and space, so that the microscopic Maxwell equations smooth to

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\langle \rho_{\text{micro}} \rangle}{\epsilon_0} && (10.6) \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && (10.7) \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 && (10.8) \\ \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \langle \mathbf{J}_{\text{micro}} \rangle + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \,, && (10.9) \end{aligned}$$

We split the microscopic charge and current densities into *bound charges*, which are bound to the atoms of the medium, and *free charges*, which are not bound. Examples of free charges include the mobile electrons in a conductor that carry the current in the presence of an applied electric field, and any additional charges that we have placed within or externally to the medium. We denote the macroscopic (smoothed) free charge and current densities by ρ_{free} and $\mathbf{J}_{\text{free}}$, and similarly ρ_{bound} and $\mathbf{J}_{\text{bound}}$ for the bound charges and currents.

10.1.1 Bound charge density

If the material is composed of atoms or molecues, with the nth centred on $\mathbf{x}_n(t)$, and each is made up of point charges $\{q_{i,n}\}$ at locations $\mathbf{r}_{i,n}(t)$ relative to $\mathbf{x}_n(t)$, we have

$$\begin{aligned} \langle \rho_{\text{micro}}(t, \mathbf{x}) \rangle &= \rho_{\text{free}}(t, \mathbf{x}) + \langle \sum_n \sum_i q_{i,n} \delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t) - \mathbf{r}_{i,n}(t)) \rangle \\ &= \rho_{\text{free}}(t, \mathbf{x}) + \underbrace{\sum_n \sum_i q_{i,n} f(\mathbf{x} - \mathbf{x}_n(t) - \mathbf{r}_{i,n}(t))}_{\rho_{\text{bound}}(t,\mathbf{x})} \,. \qquad (10.10) \end{aligned}$$

The relative locations $\mathbf{r}_{i,n}$ are small in magnitude compared to the support of the smoothing function by assumption. We can therefore Taylor expand to obtain

$$\begin{aligned}
\rho_{\text{bound}}(t,\mathbf{x}) &= \sum_n \sum_i q_{i,n} \left[f(\mathbf{x}-\mathbf{x}_n(t)) - \mathbf{r}_{i,n}(t)\cdot\boldsymbol{\nabla} f(\mathbf{x}-\mathbf{x}_n(t)) + \cdots \right] \\
&= -\sum_n \underbrace{\left(\sum_i q_{i,n}\mathbf{r}_{i,n}(t)\right)}_{\mathbf{p}_n(t)} \cdot \boldsymbol{\nabla} f(\mathbf{x}-\mathbf{x}_n(t)) + \cdots \\
&= -\boldsymbol{\nabla}\cdot\left[\sum_n \mathbf{p}_n(t) f(\mathbf{x}-\mathbf{x}_n(t))\right] + \cdots \\
&= -\boldsymbol{\nabla}\cdot \underbrace{\langle \sum_n \mathbf{p}_n(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_n(t))\rangle}_{\mathbf{P}(t,\mathbf{x})} + \cdots \, ,
\end{aligned} \tag{10.11}$$

where we have assumed that the net charge $\sum_i q_{i,n} = 0$. Here, we see that each atom/molecule may be replaced by its electric dipole moment $\mathbf{p}_n(t)$, plus higher-order moments represented by the ellipsis in the above, for the purpose of calculating the smoothed bound charge density. We have introduced the *macroscopic polarisation*,

$$\mathbf{P}(t,\mathbf{x}) \equiv \langle \sum_n \mathbf{p}_n(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_n(t))\rangle \, , \tag{10.12}$$

the divergence of which generates ρ_{bound}. In principle, there are contributions to ρ_{bound} from the higher multipoles but these are invariably negligible. For example, the contribution from electric quadrupole moments is smaller than that from electric dipoles by roughly a/l_{smooth}, where a is an atomic size and l_{smooth} is the smoothing length associated with $f(\mathbf{x})$.

As noted earlier, even if the material has non-zero molecular moments $\mathbf{p}_n(t)$ in the absence of an applied field, these will generally be randomly oriented on macroscopic scales and will not contribute to $\mathbf{P}$. (More carefully, the mean-squared contribution to $\mathbf{P}$ will go as $1/N$ where $N \gg 1$ is the number of molecules within a smoothing volume.) What matters for $\mathbf{P}$ is the perturbations to the $\mathbf{p}_n(t)$ due to the applied field. These will be coherent over scales large compared to the smoothing scale and so will not be averaged down by smoothing.

Note that we have only discussed spatial smoothing, not smoothing in time. The latter is not required once we smooth in space since the resulting $\mathbf{P}(t,\mathbf{x})$ will only vary on the timescales of the applied fields. Although intrinsic dipole moments $\mathbf{p}_n(t)$ may vary over a range of timescales, these motions are incoherent on macroscopic scales and will average out in macroscopic volumes.

Bound surface charge

We have seen that $\rho_{\text{bound}} = -\boldsymbol{\nabla}\cdot\mathbf{P}$. Generally, this will be non-zero inside a material provided the polarisation is non-uniform. But note that, even if the polarisation is uniform inside the material, there will still be bound charges on the surface of the material. To see this, note that as we move from inside the material to the external vacuum, $\mathbf{P}$ will rapidly transition to zero over the smoothing scale l_{smooth}, producing a non-zero ρ_{bound} localised near the surface. If we only care about length scales large compared to l_{smooth}, we can regard this as a surface charge density σ_{bound}, i.e., a charge per area of the surface, which we can determine from the divergence theorem:

$$\sigma_{\text{bound}} dS = \int_V \rho_{\text{bound}}\, d^3\mathbf{x} \tag{10.13}$$

$$= -\int_{\partial V} \mathbf{P}\cdot d\mathbf{S} \tag{10.14}$$

$$= \mathbf{P}\cdot d\mathbf{S}\,. \tag{10.15}$$

Here, the integral is over a "Gaussian pillbox" that straddles the transition region at the surface, and $d\mathbf{S} = \hat{\mathbf{n}} dS$ is the outward-pointing area element. We see that the surface charge is

$$\sigma_{\text{bound}} = \mathbf{P}\cdot\hat{\mathbf{n}}\,, \tag{10.16}$$

where $\mathbf{P}$ is the polarisation just inside the bulk of the material (but away from the transition region), and $\hat{\mathbf{n}}$ is the unit outward normal to the surface.

10.1.2 Bound current density

The calculation of the bound current density proceeds similarly, but is rather more involved because of the vector nature of the current and the presence of molecular velocities $\mathbf{v}_n(t) \equiv \dot{\mathbf{x}}_n$.

Generally, we have

$$\begin{aligned}\langle \mathbf{J}_{\text{micro}}(t,\mathbf{x})\rangle &= \mathbf{J}_{\text{free}}(t,\mathbf{x}) + \langle \sum_n \sum_i q_{i,n}\left[\mathbf{v}_n(t) + \mathbf{v}_{i,n}(t)\right] \delta^{(3)}(\mathbf{x}-\mathbf{x}_n(t)-\mathbf{r}_{i,n}(t))\rangle \\ &= \mathbf{J}_{\text{free}}(t,\mathbf{x}) + \underbrace{\sum_n \sum_i q_{i,n}\left[\mathbf{v}_n(t) + \mathbf{v}_{i,n}(t)\right] f(\mathbf{x}-\mathbf{x}_n(t)-\mathbf{r}_{i,n}(t))}_{\mathbf{J}_{\text{bound}}(t,\mathbf{x})}\,.\end{aligned} \tag{10.17}$$

Here, $\mathbf{v}_{i,n} \equiv \dot{\mathbf{r}}_{i,n}$ and in the limit of non-relativistic velocities is the internal relative velocity of the ith charge in the nth molecule. We shall initially ignore the effects of the molecular velocities $\mathbf{v}_n(t)$ since they are usually much smaller than the rapid internal velocities (e.g., of electrons in the molecule) and tend not to be coherent over large smoothing volumes. In this approximation, we should also ignore the time dependence of $f(\mathbf{x}-\mathbf{x}_n(t))$ since

$$\frac{\partial f(\mathbf{x}-\mathbf{x}_n(t))}{\partial t} = -\mathbf{v}_n(t)\cdot\boldsymbol{\nabla} f(\mathbf{x}-\mathbf{x}_n(t))\,. \tag{10.18}$$

We then have

$$\begin{aligned}
\mathbf{J}_{\text{bound}}(t,\mathbf{x}) &\approx \sum_n\sum_i q_{i,n}\mathbf{v}_{i,n}(t)\left[f(\mathbf{x}-\mathbf{x}_n(t)) - \mathbf{r}_{i,n}(t)\cdot\boldsymbol{\nabla} f(\mathbf{x}-\mathbf{x}_n(t)) + \cdots\right] \\
&\approx \sum_n \dot{\mathbf{p}}_n(t) f(\mathbf{x}-\mathbf{x}_n(t)) - \sum_n\sum_i q_{i,n}\mathbf{v}_{i,n}(t)\mathbf{r}_{i,n}(t)\cdot\boldsymbol{\nabla} f(\mathbf{x}-\mathbf{x}_n(t)) + \cdots \\
&\approx \frac{\partial \mathbf{P}(t,\mathbf{x})}{\partial t} - \sum_n\sum_i q_{i,n}\mathbf{v}_{i,n}(t)\mathbf{r}_{i,n}(t)\cdot\boldsymbol{\nabla} f(\mathbf{x}-\mathbf{x}_n(t)) + \cdots\,.
\end{aligned} \tag{10.19}$$

The first term in the last line is the time derivative of the macroscopic polarization, and arises since changes in the molecular dipole moments are accompanied by a flow of charge within the molecules. We can see this clearly from the continuity equation for the bound charge: given that $\rho_{\text{bound}} = -\boldsymbol{\nabla}\cdot\mathbf{P}+\cdots$, the divergence of $\mathbf{J}_{\text{bound}}$ must include the divergence of $\partial\mathbf{P}/\partial t$.

For the second term in Eq. (10.19), we use

$$\begin{aligned}
\sum_n\sum_i q_{i,n}\mathbf{v}_{i,n}(t)\mathbf{r}_{i,n}(t)\cdot\boldsymbol{\nabla} f(\mathbf{x}-\mathbf{x}_n(t)) &\approx \frac{1}{2}\sum_n\sum_i q_{i,n}\mathbf{v}_{i,n}(t)\mathbf{r}_{i,n}(t)\cdot\boldsymbol{\nabla} f(\mathbf{x}-\mathbf{x}_n(t)) \\
&\quad - \frac{1}{2}\sum_n\sum_i q_{i,n}\mathbf{r}_{i,n}(t)\mathbf{v}_{i,n}(t)\cdot\boldsymbol{\nabla} f(\mathbf{x}-\mathbf{x}_n(t)) \\
&\quad + \frac{1}{2}\frac{\partial}{\partial t}\sum_n\sum_i q_{i,n}\mathbf{r}_{i,n}(t)\mathbf{r}_{i,n}(t)\cdot\boldsymbol{\nabla} f(\mathbf{x}-\mathbf{x}_n(t))
\end{aligned} \tag{10.20}$$

where the approximation is to ignore the time dependence of $f(\mathbf{x}-\mathbf{x}_n(t))$. (This is similar to the rearrangement in Eq. 6.23.) The first two terms on the right combine to give

$$\begin{aligned}
-\boldsymbol{\nabla}\times\left(\frac{1}{2}\sum_n\sum_i q_{i,n}\mathbf{r}_{i,n}(t)\times\mathbf{v}_{i,n}(t) f(\mathbf{x}-\mathbf{x}_n(t))\right) &= -\boldsymbol{\nabla}\times\sum_n \mathbf{m}_n(t) f(\mathbf{x}-\mathbf{x}_n(t)) \\
&= -\boldsymbol{\nabla}\times\underbrace{\langle\sum_n \mathbf{m}_n(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_n(t))\rangle}_{\mathbf{M}(t,\mathbf{x})}\,,
\end{aligned} \tag{10.21}$$

where $\mathbf{m}_n$ is the molecular *magnetic dipole moment*,

$$\mathbf{m}_n(t) \equiv \frac{1}{2} \sum_i q_{i,n} \mathbf{r}_{i,n} \times \mathbf{v}_{i,n} \,, \tag{10.22}$$

and $\mathbf{M}(t, \mathbf{x})$ is the *macroscopic magnetisation.* The final term on the right of Eq. (10.20) involves the *molecular quadrupole moment* tensor,

$$\mathsf{Q}'_n(t) = 3 \sum_i q_{i,n} \mathbf{r}_{i,n}(t) \otimes \mathbf{r}_{i,n}(t) \,. \tag{10.23}$$

We have

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \sum_n \sum_i q_{i,n} \mathbf{r}_{i,n}(t) \mathbf{r}_{i,n}(t) \cdot \boldsymbol{\nabla} f(\mathbf{x} - \mathbf{x}_n(t)) &= \frac{1}{6} \frac{\partial}{\partial t} \sum_n \mathsf{Q}'_n(t) \boldsymbol{\nabla} f(\mathbf{x} - \mathbf{x}_n(t)) \\ &= \frac{1}{6} \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \underbrace{\langle \sum_n \mathsf{Q}'_n(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t)) \rangle}_{\mathsf{Q}'(t,\mathbf{x})} \,, \end{aligned} \tag{10.24}$$

where $\mathsf{Q}(t, \mathbf{x})$ is the *macroscopic quadrupole density.* Note that $\boldsymbol{\nabla} \cdot \mathsf{Q}'$ is shorthand for the vector with components $\partial Q'_{ij} / \partial x_j$. Note further that the quadrupole moments here have not had the trace removed; we denote these with an additional prime to distinguish them from the trace-free quadrupole moments, e.g., Eq. (6.26).

Putting these pieces together, we finally find

$$\mathbf{J}_{\text{bound}} \approx \frac{\partial \mathbf{P}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{M} - \frac{1}{6} \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathsf{Q}' + \cdots \,. \tag{10.25}$$

We have already discussed the physical origin of the first term on the right, but what about the others? The third (i.e., quadrupole) term arises for the same reason as the first: if we include higher terms in the expansion of ρ_{bound} in Eq. (10.11), the next term involves the divergence of $\boldsymbol{\nabla} \cdot \mathsf{Q}'$:

$$\rho_{\text{bound}}(t, \mathbf{x}) = -\boldsymbol{\nabla} \cdot \left(\mathbf{P}(t, \mathbf{x}) - \frac{1}{6} \boldsymbol{\nabla} \cdot \mathsf{Q}'(t, \mathbf{x}) + \cdots \right) \,. \tag{10.26}$$

The bound current must include the time derivative of the term in brackets to ensure charge conservation. However, we can also add divergence-free terms to $\mathbf{J}_{\text{bound}}$ since these would not affect the continuity equation. We see that the first such term is the curl of the magnetisation. One can think of the molecular dipole moments as loops of current; if these vary spatially, adjacent loops will not cancel perfectly and can give rise to a net bound current density in the bulk of the material. Often only the first two terms are retained in Eq. (10.25). As noted above, the quadrupole term is typically smaller by a/l_{smooth} than the electric polarization. The quadrupole and magnetisation terms are each of the same order in $a/l_{\rm smooth}$ and the internal relative velocities, but retaining the magnetisation can be important in some situations, for example magnetostatics where the magnetic field induces magnetic dipole moments but no electric multipole moments.

There is one loose end to tidy up – the neglect of the molecular velocities $\mathbf{v}_n(t)$. It is left as an exercise to show that in this case there are additional contributions to the bound current that involve products of the $\mathbf{v}_n(t)$ with the molecular electric dipole and quadrupole moments. As noted above, such terms are generally negligible. An exception is if the medium is in relativistic bulk motion in which case the $\mathbf{v}_n(t)$ are coherent over a smoothing volume. This case can also be dealt with directly by Lorentz transformation.

Bound surface current

In the same way that the rapid variation in $\mathbf{P}$ at the surface of the material gives rise to a bound surface charge density, surface currents flow at the surface due to the rapid variation in $\mathbf{M}$. We can determine the surface current density[12] $\mathbf{K}_{\rm bound}$ by integrating Eq. (10.25) across an infinitesimal plane surface whose normal is tangent to the surface of the material. The boundary of the infinitesimal surface is a planar circuit that straddles the material surface. The long sides of this circuit are tangent to the surface of the material and have length Δl. One of the long sides lies just inside the material and is $\Delta\mathbf{l}$, and the other ($-\Delta\mathbf{l}$) is just outside. The short sides have negligible length. Let the surface of the material have outward normal $\hat{\mathbf{n}}$, and the normal to the circuit be $\hat{\mathbf{t}}$. It follows that $\Delta\mathbf{l} = \Delta l \hat{\mathbf{n}} \times \hat{\mathbf{t}}$. The integral of $\boldsymbol{\nabla} \times \mathbf{M}$ across the surface that spans the circuit gives a contribution to the surface current; by Stokes theorem,

$$\begin{aligned} \int (\boldsymbol{\nabla} \times \mathbf{M}) \cdot d\mathbf{S} &= \oint \mathbf{M} \cdot d\mathbf{l} \\ &= \mathbf{M} \cdot \Delta\mathbf{l}\,, \end{aligned} \tag{10.27}$$

where $\mathbf{M}$ is the magnetisation just inside the material (but away from the transition region). If the corresponding surface current is $\mathbf{K}_{\rm bound}$, we must have

$$\begin{aligned} & \mathbf{M} \cdot \Delta\mathbf{l} = \mathbf{K}_{\rm bound} \cdot (\hat{\mathbf{t}}\Delta l) \\ \Rightarrow \qquad & \hat{\mathbf{t}} \cdot (\mathbf{M} \times \hat{\mathbf{n}}) = \hat{\mathbf{t}} \cdot \mathbf{K}_{\rm bound} \qquad \forall\, \hat{\mathbf{t}} \perp \hat{\mathbf{n}}\,. \end{aligned} \tag{10.28}$$

It follows that

$$\mathbf{K}_{\rm bound} = \mathbf{M} \times \hat{\mathbf{n}}\,. \tag{10.29}$$

There is no further contribution to the surface current from the $\partial\mathbf{P}/\partial t$ term in Eq. (10.25) since the spanning surface has vanishing area. There is, however, in general a further contribution from the quadrupole term in Eq. (10.25).

[12]Generally, a surface current density is represented by a vector $\mathbf{K}$ lying in the surface. The direction represents the direction the current flows, and the magnitude gives the current per length through a line in the surface that is perpendicular to $\mathbf{K}$.

As a simple application of Eq. (10.29), consider a uniformly magnetised rod, with $\mathbf{M}$ pointing along the rod. This is a simple model for a permanent bar magnet. The magnetisation is uniform inside the rod, so the only current is a surface current that flows uniformly around the rod. The magnetic field is therefore equivalent to that of a finite solenoid with a current per length $|\mathbf{M}|$.

10.2 Electric displacement D

Let us now substitute the free and bound charge density into the macroscopic Maxwell equation $\nabla \cdot \mathbf{E} = \langle \rho_{\text{micro}} \rangle / \epsilon_0$. Retaining only the contribution from the macroscopic polarisation to the bound charge, we have

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0} \left(\rho_{\text{free}} - \boldsymbol{\nabla} \cdot \mathbf{P} \right) . \tag{10.30}$$

If we define the *electric displacement*,

$$\mathbf{D}(t, \mathbf{x}) = \epsilon_0 \mathbf{E}(t, \mathbf{x}) + \mathbf{P}(t, \mathbf{x}) \,, \tag{10.31}$$

we obtain the simpler-looking equation

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_{\text{free}} \,. \tag{10.32}$$

Of course, the complications of being inside a medium have not disappeared: they have been shifted to the relation between $\mathbf{D}$ and the macroscopic electric field $\mathbf{E}$.

As discussed earlier, it is often a good approximation to assume that the induced dipole moments are linear in the electric field. If the medium is also isotropic, then we must have[13]

$$\mathbf{P}(t, \mathbf{x}) = \chi_e \epsilon_0 \mathbf{E}(t, \mathbf{x}) \,, \tag{10.33}$$

where the dimensionless constant χ_e is the *electrical susceptibility* of the medium. We can then write

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} \,, \tag{10.34}$$

where $\epsilon_r = 1 + \chi_e$ is the *relative permittivity* of the medium. For dilute gases, ϵ_r is close to unity, but in strongly polar substances it can be much larger (e.g., $\epsilon_r \approx 80$ for water at room temperature).

Inside a homogeneous medium, ϵ_r is uniform and so Eq. (10.32) reduces to

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho_{\text{free}}}{\epsilon_0 \epsilon_r} \,. \tag{10.35}$$

[13]This is actually one simplification too far. More generally, the polarization will be local in space, but non-local in time, i.e., $\mathbf{P}(t, \mathbf{x})$ will depend on $\mathbf{E}(t', \mathbf{x})$ at all times $t' \leq t$. We shall discuss such *dispersive* media later in Sec. 12.

Combining this with the macroscopic Maxwell equation $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, we see that electrostatics inside an infinite such medium amounts to reducing the fields that would have been generated by the same free charges in vacuum by a factor ϵ_r. The origin of this reduction is a screening of the free charges by the bound charges that arise from the divergence of the induced polarization.

Example: a dielectric sphere with free charge Q at the centre. Consider a uniform and isotropic dielectric sphere of radius R and relative permittivity ϵ_r. Let a free charge Q be placed at its centre. Inside the sphere, Eq. (10.32) implies that

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{x}} \qquad (r < R)\,. \tag{10.36}$$

There are no free charges outside the sphere, so $\nabla\cdot\mathbf{D} = 0$ there. The boundary (or junction) condition on $\mathbf{D}$ follows from integrating $\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_{\text{free}}$ over a Gaussian pillbox at the surface. Since there is no free charge there, the perpendicular component of $\mathbf{D}$ is continous, so that Eq. (10.36) also holds for $r > R$.

The electric field follows easily from $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$. Inside the sphere, we have

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0\epsilon_r r^2}\hat{\mathbf{x}} \qquad (r < R)\,. \tag{10.37}$$

This is indeed smaller than the value in vacuum by a factor of ϵ_r. Since the polarization $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$ is divergence-free for $0 < r < R$, the only bound charges are at the centre and the surface of the sphere. The bound charge at the centre must combine with the free charge Q to give the electric field in Eq. (10.37), so that

$$\begin{aligned} Q_{\text{bound}} + Q &= Q/\epsilon_r \\ \Rightarrow \qquad Q_{\text{bound}} &= Q\left(\frac{1-\epsilon_r}{\epsilon_r}\right)\,. \end{aligned} \tag{10.38}$$

The surface charge at $r = R$ follows from $\sigma_{\text{bound}} = \mathbf{P}\cdot\hat{\mathbf{n}}$, and so

$$4\pi R^2\sigma_{\text{bound}} = Q\left(\frac{\epsilon_r - 1}{\epsilon_r}\right)\,, \tag{10.39}$$

where we have used $\chi_e = \epsilon_r - 1$. It follows that the total bound charge is zero, as it must be since the dielectric is electrically neutral.

Finally, the electric field outside the sphere is simply

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{x}} \qquad (r > R)\,. \tag{10.40}$$

This is the same as if the sphere were not there, a consequence of the spherical symmetry and the vanishing of the total bound charge. Note that the perpendicular component of $\mathbf{E}$ is discontinuous due to the presence of the surface charge.

10.3 Magnetising field H

We now substitute $\langle \mathbf{J}_{\text{micro}} \rangle = \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{bound}}$ into the averaged Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \langle \mathbf{J}_{\text{micro}} \rangle + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} . \tag{10.41}$$

Retaining only the contributions to the bound current from electric and magnetic dipoles, we have

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}_{\text{free}} + \mu_0 \left(\nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\ \Rightarrow \nabla \times (\mathbf{B} - \mu_0 \mathbf{M}) &= \mu_0 \mathbf{J}_{\text{free}} + \mu_0 \frac{\partial}{\partial t} (\epsilon_0 \mathbf{E} + \mathbf{P}) . \end{aligned} \tag{10.42}$$

If we introduce the *magnetising field* **H** with

$$\mu_0 \mathbf{H}(t, \mathbf{x}) \equiv \mathbf{B}(t, \mathbf{x}) - \mu_0 \mathbf{M}(t, \mathbf{x}) , \tag{10.43}$$

and recalling the electric displacement $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$, we can write

$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t} . \tag{10.44}$$

We see that the magnetising field plays a similar role to the electric displacement, in that the bound currents due to the magnetisation are absorbed into **H**. Note also how the contribution to the bound current from $\partial \mathbf{P}/\partial t$ got absorbed into $\partial \mathbf{D}/\partial t$, leaving only the free current as a source.

In linear, isotropic media the magnetisation can be written as

$$\mu_0 \mathbf{M}(t, \mathbf{x}) = \frac{\chi_m}{1 + \chi_m} \mathbf{B}(t, \mathbf{x}) , \tag{10.45}$$

where χ_m is the dimensionless *magnetic susceptibility.* It follows that **B** and **H** are also linearly related,

$$\mathbf{B}(t, \mathbf{x}) = \mu_0 \mu_r \mathbf{H}(t, \mathbf{x}) , \tag{10.46}$$

where the *relative permeability* is

$$\mu_r = 1 + \chi_m . \tag{10.47}$$

[The simplicity of this relation is reason for the rather odd-looking way that χ_m is introduced in Eq. (10.45).] The magnetic susceptibility is typically $\chi_m \sim O(10^{-5})$. For diamagnets (**M** antiparallel to **B**), $-1 < \chi_m < 0$ and so $0 < \mu_r < 1$; for paramagnets (**M** parallel to **B**), $\chi_m > 0$ and so $\mu_r > 1$. For ferromagnets, the relation between **B** and **H** is non-linear and depends on the history of preparation of the material.

Finally, we gather together the *macroscopic Maxwell equations* for the averaged fields **E** and

$\mathbf{B}$ and their associated fields $\mathbf{D}$ and $\mathbf{H}$ inside continuous media:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{D} &= \rho_{\text{free}} \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{H} &= \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t} \,.
\end{aligned} \tag{10.48}$$

These can be solved given the free charges and currents once the *constitutive relations* relating $\mathbf{D}$ and $\mathbf{H}$ to $\mathbf{E}$ and $\mathbf{B}$ are specified. These are dependent on the type of material; in the simplest linear, isotropic materials they are fully specified by two constants, ϵ_r and μ_r, with

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} \qquad \text{and} \qquad \mathbf{B} = \mu_0 \mu_r \mathbf{H} \,. \tag{10.49}$$

Later, in Sec. 12, we shall briefly consider more complicated relations where the relation between e.g., $\mathbf{D}$ and $\mathbf{E}$ is still linear and local in space, but non-local in time.

11 Electromagnetic waves in simple dielectric media

Maxwell's equations admit wavelike solutions in vacuum that propagate at the speed of light. In this section we extend this idea to the propagation of electromagnetic waves in simple linear, isotropic dielectric media. We shall also consider how waves reflect off boundaries between two such materials.

For the simple constitutive relations in Eq. (10.49), with ϵ_r and μ_r constants, Maxwell's equations in the absence of any free charges or currents reduce to

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{B} &= \frac{\epsilon_r \mu_r}{c^2} \frac{\partial \mathbf{E}}{\partial t} \,.
\end{aligned} \tag{11.1}$$

Taking the curl of the second and fourth equations, and using that $\mathbf{E}$ and $\mathbf{B}$ are divergence-free, we recover wave equations:

$$\nabla^2 \mathbf{E} = \frac{\epsilon_r \mu_r}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \text{and} \qquad \nabla^2 \mathbf{B} = \frac{\epsilon_r \mu_r}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} \,. \tag{11.2}$$

The only effect of the medium is to change the propagation speed from c to

$$v = \frac{c}{\sqrt{\epsilon_r \mu_r}} \,. \tag{11.3}$$

The wave speed in the medium is often written in terms of the *index of refraction* n, with $v = c/n$, so that

$$n = \sqrt{\epsilon_r \mu_r}\,. \tag{11.4}$$

As we have seen, in most materials $\mu_r \approx 1$, but ϵ_r can be large making v rather less than c.

We can look for plane-wave solutions of the wave equations with wavevector $\mathbf{k}$ and (angular) frequency ω:

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \qquad \text{and} \qquad \mathbf{B}(t,\mathbf{x}) = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,, \tag{11.5}$$

where it is implicit that the fields are the real part of the complex expressions on the right-hand sides. Here, $\mathbf{E}_0$ and $\mathbf{B}_0$ are constant, but generally complex, polarization vectors. The fields are divergence-free, which requires $\mathbf{k}\cdot\mathbf{E}_0 = 0$ and $\mathbf{k}\cdot\mathbf{B}_0 = 0$. To satisfy the wave equations (11.2), ω and $\mathbf{k}$ must be related by the *dispersion relation*

$$\omega^2 = v^2 \mathbf{k}^2\,. \tag{11.6}$$

Finally, to satisfy $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, the polarization vectors must be related by

$$\mathbf{B}_0 = \mathbf{k}\times\mathbf{E}_0/\omega\,. \tag{11.7}$$

If all components of the wavevector are real[14] we have $\mathbf{B}_0 = \hat{\mathbf{k}}\times\mathbf{E}_0/v$.

Finally, note that care is needed when forming products of fields when the fields are represented by the real part of complex quantities. For example,

$$\begin{aligned}\mathbf{E}\times\mathbf{B} &= \Re\left(\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)\times\Re\left(\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)\\ &= \frac{1}{2}\Re\left(\mathbf{E}_0\times\mathbf{B}_0 e^{2i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \mathbf{E}_0\times\mathbf{B}_0^*\right)\,,\end{aligned} \tag{11.8}$$

where we used the identity $\Re(z_1)\Re(z_2) = \Re(z_1 z_2 + z_1 z_2^*)/2$ for any complex quantities z_1 and z_2. If we are only interested in time-averaged products of fields (which is generally the case when discussing net flows of energy and momentum), we have simply $\langle\mathbf{E}\times\mathbf{B}\rangle_t = \Re(\mathbf{E}_0\times\mathbf{B}_0^*)/2$.

Boundary conditions

The wave solutions given above are valid in uniform media, but we shall also be interested in situations where the material properties are only piecewise constant. In this case, we have boundaries between regions with different ϵ_r and μ_r, and we need to know how the fields are related on either side of a boundary in order to patch our wave solutions for the fields together.

We shall only consider the case where these are no *free* surface charges or currents. The junction conditions then follow from integrating Maxwell's equations (10.48), with ρ_{free} and $\mathbf{J}_{\text{free}}$ both zero, across the boundary in the usual way (see IB *Electromagnetism*). We start

[14] This need not be the case; we shall meet examples shortly where some components of $\mathbf{k}$ are imaginary.

with $\nabla \cdot \mathbf{D} = 0$ and $\nabla \cdot \mathbf{B} = 0$. Integrating these over a "Gaussian pillbox" across the surface (whose unit normal is $\hat{\mathbf{n}}$) gives

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0 \tag{11.9}$$

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0\,, \tag{11.10}$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are the electric displacements either side of the surface, and similarly for $\mathbf{B}_1$ and $\mathbf{B}_2$. In words, *the normal component of the electric displacement and the normal component of the magnetic field are continuous across the boundary.* Generally, this will mean that the normal component of the electric field is discontinuous. The discontinuity is sourced by *bound* surface charge due to the different macroscopic polarizations $\mathbf{P}$ on either side of the boundary.

For the parallel field components, we integrate around an infinitesimal circuit with long sides parallel to and on either side of the boundary. This gives

$$\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{11.11}$$

$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0\,. \tag{11.12}$$

In words, *the parallel components of the electric field and magnetising field are continuous across the boundary.* Note that, generally, the parallel components of $\mathbf{B}$ will be discontinuous due to the bound surface current that flows at the surface between two media with different magnetisations $\mathbf{M}$.

11.1 Reflection and refraction

We now consider the reflection of waves from the plane boundary between two semi-infinite, linear and isotropic dielectrics. We take the boundary surface to be $x = 0$ and consider a plane wave incident from $x < 0$. We shall set $\mu_r = 1$ everywhere, but allow ϵ_r to change discontinuously from $\epsilon_{r,I}$ $(x < 0)$ to $\epsilon_{r,T}$ $(x > 0)$. There is an associated change in refractive index from n_I to n_T.

The incident wave will generally scatter of the boundary, giving rise to a reflected wave and a transmitted wave. These must have the same frequency ω as the incident wave if we are to satisfy the boundary conditions for all times. We take the incident wavevector to lie in the x-y plane at an *angle of incidence* θ_I to the x-direction:

$$\mathbf{k}_I = k_I \left(\cos\theta_I \hat{\mathbf{x}} + \sin\theta_I \hat{\mathbf{y}}\right)\,, \tag{11.13}$$

where $k_I = \omega/v_I$ and the wave speed $v_I = c/n_I = c/\sqrt{\epsilon_{r,I}}$. The electric field of the incident wave is then

$$\mathbf{E}_{\text{inc}}(t, \mathbf{x}) = \mathbf{E}_I e^{i(\mathbf{k}_I \cdot \mathbf{x} - \omega t)}\,. \tag{11.14}$$

We similarly write the reflected wave as

$$\mathbf{E}_{\text{ref}}(t, \mathbf{x}) = \mathbf{E}_R e^{i(\mathbf{k}_R \cdot \mathbf{x} - \omega t)}\,, \tag{11.15}$$

and the transmitted wave as

$$\mathbf{E}_{\text{trans}}(t, \mathbf{x}) = \mathbf{E}_T e^{i(\mathbf{k}_T \cdot \mathbf{x} - \omega t)} \,. \tag{11.16}$$

Note that $\omega^2 = v_I^2 \mathbf{k}_R^2 = v_T^2 \mathbf{k}_T^2$. The magnetic fields follow from Eq. (11.7). Note, finally, that the total electric field for $x < 0$ is given by the sum $\mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}$, while the field for $x > 0$ is simply $\mathbf{E}_{\text{trans}}$.

We need to apply the boundary conditions (11.9–11.12) at all points in the plane $x = 0$ and for all times. The only way we can satisfy the boundary conditions is if all three phase factors are equal for all points in the plane, i.e.,

$$\mathbf{k}_I \cdot \mathbf{x} = \mathbf{k}_R \cdot \mathbf{x} = \mathbf{k}_T \cdot \mathbf{x} = 0 \qquad \text{for } x = 0 \,. \tag{11.17}$$

It follows that the projections of all three wavevectors into the surface are equal, and so the reflected and transmitted wavevectors also lie in the x-y plane. Writing

$$\mathbf{k}_R = k_I \left(-\cos\theta_R \hat{\mathbf{x}} + \sin\theta_R \hat{\mathbf{y}}\right) \tag{11.18}$$

$$\mathbf{k}_T = k_T \left(\cos\theta_T \hat{\mathbf{x}} + \sin\theta_T \hat{\mathbf{y}}\right) \,, \tag{11.19}$$

we have

$$k_I \sin\theta_I = k_I \sin\theta_R = k_T \sin\theta_T \,. \tag{11.20}$$

It follows that

$$\theta_I = \theta_R \,, \tag{11.21}$$

so that the angle of incidence equals the angle of reflection. If we further use $\omega = v_I k_I = v_T k_T$, Eq. (11.20) implies that

$$\frac{\sin\theta_I}{v_I} = \frac{\sin\theta_T}{v_T} \qquad \text{or equivalently} \qquad n_I \sin\theta_I = n_T \sin\theta_T \,. \tag{11.22}$$

This is the law of refraction, known as *Snell's law.*

11.2 Fresnel equations

We now look at the amplitudes and phases of the reflected and transmitted waves. It is simplest to consider separately the two possible linear polarizations of the incident wave: $\mathbf{E}_I$ normal to the x-y plane and $\mathbf{E}_I$ parallel to the x-y plane. The general case can be treated as a linear combination of these.

11.2.1 Normal polarization

For normal polarization, the electric field of the incident wave is normal to the plane of incidence spanned by $\mathbf{k}_I$ and the normal to the surface. In our set-up, this means we take $\mathbf{E}_I = E_I \hat{\mathbf{z}}$ for some complex amplitude E_I. The electric fields of the reflected and transmitted waves are necessarily also along $\hat{\mathbf{z}}$. Continuity of the perpendicular component of $\mathbf{D}$ is therefore identically satisfied, while continuity of the parallel component of $\mathbf{E}$ requires[15]

$$E_I + E_R = E_T \,. \tag{11.23}$$

The magnetic fields follow from Eq. (11.7); we have

$$\mathbf{B}_I = \frac{E_I}{v_I}\left(\sin\theta_I\hat{\mathbf{x}} - \cos\theta_I\hat{\mathbf{y}}\right) \tag{11.24}$$
$$\mathbf{B}_R = \frac{E_R}{v_I}\left(\sin\theta_I\hat{\mathbf{x}} + \cos\theta_I\hat{\mathbf{y}}\right) \tag{11.25}$$
$$\mathbf{B}_T = \frac{E_T}{v_T}\left(\sin\theta_T\hat{\mathbf{x}} - \cos\theta_T\hat{\mathbf{y}}\right)\,, \tag{11.26}$$

where we have used $\theta_R = \theta_I$. Continuity of the perpendicular component of $\mathbf{B}$ gives

$$\frac{E_I + E_R}{v_I}\sin\theta_I = \frac{E_T}{v_T}\sin\theta_T\,, \tag{11.27}$$

but with Snell's law this just reproduces $E_I + E_R = E_T$. However, a new constraint does arise from continuity of the parallel component of $\mathbf{H}$ (which, here, means the parallel component of $\mathbf{B}$ is continuous since we are taking $\mu_r = 1$):

$$\frac{E_I - E_R}{v_I}\cos\theta_I = \frac{E_T}{v_T}\cos\theta_T\,. \tag{11.28}$$

Combining Eqs (11.27) and (11.28), and expressing the velocities in terms of the refractive indices, we find

$$\frac{E_R}{E_I} = \frac{n_I\cos\theta_I - n_T\cos\theta_T}{n_I\cos\theta_I + n_T\cos\theta_T} \tag{11.29}$$
$$\frac{E_T}{E_I} = \frac{2n_I\cos\theta_I}{n_I\cos\theta_I + n_T\cos\theta_T}\,. \tag{11.30}$$

Note that $n_T\cos\theta_T$ can be related to $\sin\theta_I$ via Snell's law:

$$n_T\cos\theta_T = \sqrt{n_T^2 - n_I^2\sin^2\theta_I}\,. \tag{11.31}$$

The amplitude ratios (11.29) and (11.30) are plotted in Fig. 1 for $n_I = 1$ and $n_T = 1.5$, corresponding to a wave incident from air onto glass.

[15]If we retain components of the reflected and transmitted electric fields in the plane of incidence in the calculation, we have two further non-trivial boundary conditions from continuity of $\hat{\mathbf{n}}\cdot\mathbf{D}$ and $\hat{\mathbf{n}}\times\mathbf{E}$. These can only be satisfied if the electric field components of both waves in the plane of incidence vanish.

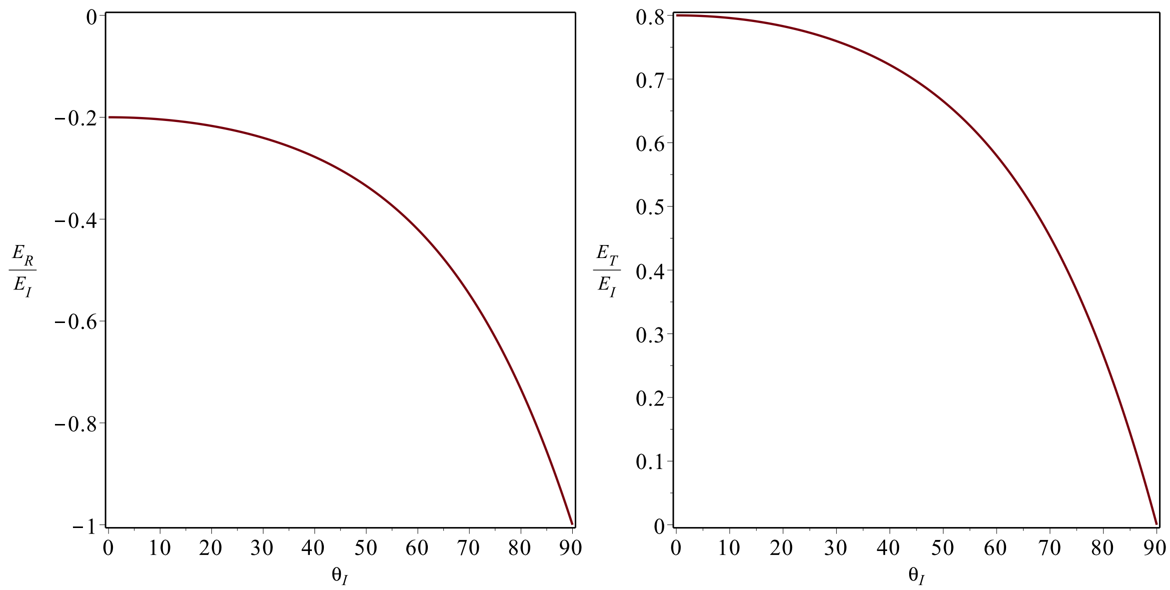

Figure 1: Ratio E_R/E_I (left) and E_T/E_I (right) as a function of the angle of incidence θ_I for light with normal polarization incident on the plane boundary between media with $n_I = 1$ and $n_T = 1.5$.

11.2.2 Parallel polarization

In the case of parallel polarization, the incident electric field is parallel to the plane of incidence, so in our set-up lies in the x-y plane. The same will be true of the reflected and transmitted electric fields, so we can can write

$$\mathbf{E}_I = E_I \left(\sin\theta_I \hat{\mathbf{x}} - \cos\theta_I \hat{\mathbf{y}}\right) \tag{11.32}$$

$$\mathbf{E}_R = E_R \left(\sin\theta_I \hat{\mathbf{x}} + \cos\theta_I \hat{\mathbf{y}}\right) \tag{11.33}$$

$$\mathbf{E}_T = E_T \left(\sin\theta_T \hat{\mathbf{x}} - \cos\theta_T \hat{\mathbf{y}}\right) \,. \tag{11.34}$$

The magnetic field of the three waves are all along the z-direction, with

$$\mathbf{B}_I = -\frac{E_I}{v_I}\hat{\mathbf{z}}\,, \qquad \mathbf{B}_R = -\frac{E_R}{v_I}\hat{\mathbf{z}}\,, \qquad \mathbf{B}_T = -\frac{E_T}{v_T}\hat{\mathbf{z}}\,. \tag{11.35}$$

Continuity of the perpendicular component of $\mathbf{B}$ is automatically satisfied, while continuity of the parallel component of $\mathbf{H}$ gives (recall, $\mu_r = 1$ here)

$$\frac{E_I + E_R}{v_I} = \frac{E_T}{v_T}\,. \tag{11.36}$$

For the electric field, continuity of the parallel component of $\mathbf{E}$ gives

$$E_I \cos\theta_I - E_R \cos\theta_I = E_T \cos\theta_T \,, \tag{11.37}$$

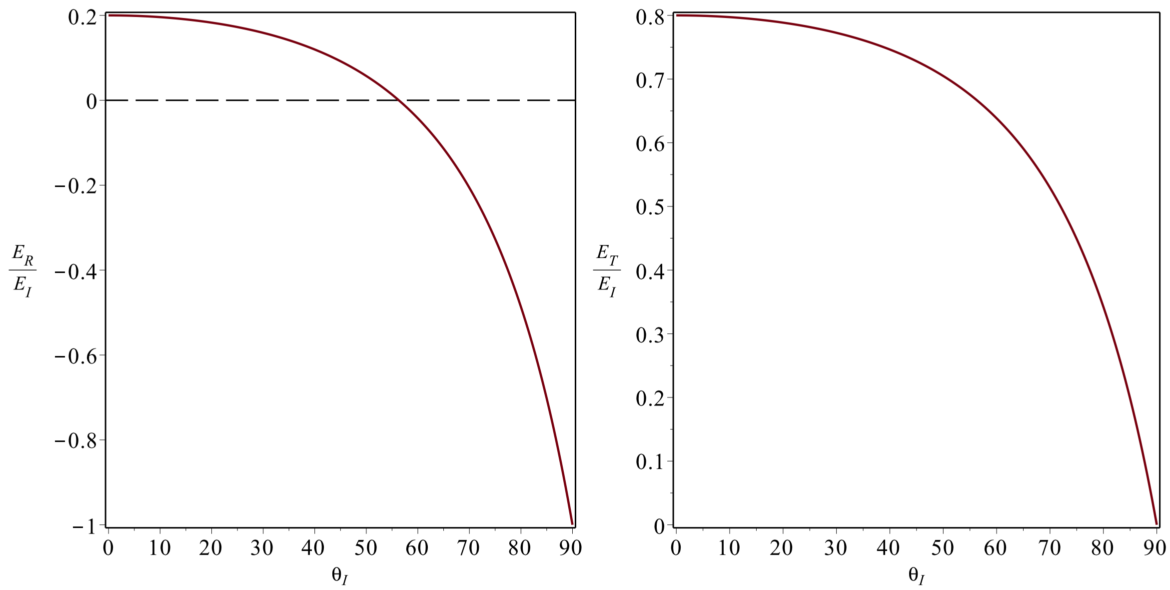

Figure 2: Ratio E_R/E_I (left) and E_T/E_I (right) as a function of the angle of incidence θ_I for light with parallel polarization incident on the plane boundary between media with $n_I = 1$ and $n_T = 1.5$. Here, Brewster's angle, where $E_R = 0$, is $\theta_\mathrm{B} = 56.3°$.

while continuity of the perpendicular component of **D** requires

$$\frac{(E_I + E_R)\sin\theta_I}{v_I^2} = \frac{E_T \sin\theta_T}{v_T^2}\,. \tag{11.38}$$

This latter equation is equivalent to Eq. (11.36) on using Snell's law. Combining Eqs (11.36) and (11.37), we find the amplitude ratios

$$\frac{E_R}{E_I} = \frac{n_T\cos\theta_I - n_I\cos\theta_T}{n_I\cos\theta_T + n_T\cos\theta_I} \tag{11.39}$$

$$\frac{E_T}{E_I} = \frac{2n_I\cos\theta_I}{n_I\cos\theta_T + n_T\cos\theta_I}\,. \tag{11.40}$$

The are plotted in Fig. 2 for $n_I = 1$ and $n_T = 1.5$.

Note that for normal incidence ($\theta_I = 0$), the amplitude ratios for both normal and parallel polarization reduce to

$$\frac{E_R}{E_I} = \pm\frac{n_I - n_T}{n_I + n_T} \tag{11.41}$$

$$\frac{E_T}{E_I} = \frac{2n_I}{n_I + n_T}\,, \tag{11.42}$$

where the plus sign in E_R/E_I is for normal polarization and the negative sign is for parallel polarization.

11.2.3 Brewster's angle

It can be seen from Fig. 2 that for parallel polarization there is some value of θ_I for which there is no reflected wave and everything is transmitted. This is called *Brewster's angle*, $\theta_{\rm B}$. To determine $\theta_{\rm B}$, we note from Eq. (11.39) that $E_R = 0$ requires

$$n_T \cos\theta_I = n_I \cos\theta_T \,. \tag{11.43}$$

Combining this with Snell's law, $n_I \sin\theta_I = n_T \sin\theta_T$, we see that we must have

$$\sin\theta_I \cos\theta_I - \sin\theta_T \cos\theta_T = 0 \qquad \Rightarrow \qquad \sin 2\theta_I = \sin 2\theta_T \,, \tag{11.44}$$

so that $\theta_I + \theta_T = \pi/2$. It follows that $\cos\theta_T = \sin\theta_I$ and so, from Eq. (11.43), the Brewster angle is given by

$$\tan\theta_{\rm B} = n_T/n_I \,. \tag{11.45}$$

As an example, for the transmission from air to glass ($n_T \approx 1.5$), the Brewster angle is $\theta_{\rm B} \approx 56.3°$.

There is no such angle for which the reflected wave vanishes in the case of normal polarization[16]. If unpolarized radiation (i.e., an incoherent mix of normal and parallel polarization) is incident at the Brewster angle, the reflected radiation will be linearly polarized with its electric field normal to the plane of incidence.

There is a simple physical way to understand the condition $\theta_I + \theta_T = \pi/2$ required for no reflection. Consider an incident wave in vacuum. The reflected wave is really the radiation produced by the oscillating electric dipole moments induced in the transmitting material by the incident wave. Each molecule produces electric dipole radiation as an outgoing spherical wave with its characteristic angular distribution. However, when we combine coherently the radiation from the very large number of dipoles, the spherical waves only constructively interfere along a direction at angle θ_I to the normal and so give a plane wave propagating in the vacuum region at angle θ_I. (The transmitted wave is the superposition of the radiation field and the field of the incident wave.) For electric dipole radiation, we have seen that there is no radiation along the axis of the dipole, which, for parallel polarization, is in the plane of incidence and normal to the direction of the transmitted wavevector $\mathbf{k}_T$. It follows that if $\theta_I + \theta_T = \pi/2$, the direction along which the dipole radiation constructively interferes is the same as the axis of the dipoles and so there is no reflected wave.

[16]To see this, note from Eq. (11.29) that $E_R = 0$ would require $n_I \cos\theta_I = n_T \cos\theta_T$. Combining with Snell's law gives $\tan\theta_I = \tan\theta_T$ so that there can be no reflection only if $\theta_I = \theta_T$. This is only possible if the refractive indices of the two materials are the same, in which case the lack of reflection is trivial.

11.2.4 Total internal reflection

Consider the case where $n_I > n_T$, so $\theta_T > \theta_I$ by Snell's law. For $\sin\theta_I > n_T/n_I$, we have $\sin\theta_T > 1$. How should we interpret this?

To do so, note that

$$\begin{aligned}
\mathbf{k}_T &= k_{T,x}\hat{\mathbf{x}} + k_{T,y}\hat{\mathbf{y}} \\
&= \left(\frac{\omega^2 n_T^2}{c^2} - k_{I,y}^2\right)^{1/2}\hat{\mathbf{x}} + k_{I,y}\hat{\mathbf{y}} \qquad \text{(Snell's law)} \\
&= \frac{\omega}{c}\left(n_T^2 - n_I^2\sin^2\theta_I\right)^{1/2}\hat{\mathbf{x}} + \frac{\omega}{c}n_I\sin\theta_I\hat{\mathbf{y}}\,. \qquad (11.46)
\end{aligned}$$

The $\hat{\mathbf{x}}$ component is imaginary for $\sin\theta_I > n_T/n_I$, i.e., $\theta_I > \theta_C$ where the *critical angle of incidence* satisfies

$$\sin\theta_C = n_T/n_I\,. \qquad (11.47)$$

In this case, let us write

$$\frac{\omega}{c}\sqrt{n_T^2 - n_I^2\sin^2\theta_I} = \pm i\kappa\,, \qquad (11.48)$$

where κ is real and positive. Then

$$e^{i(\mathbf{k}_T\cdot\mathbf{x}-\omega t)} = e^{\mp\kappa x}e^{i(\omega n_I\sin\theta_I y/c-\omega t)}\,, \qquad (11.49)$$

where we should choose $-\kappa$ to ensure the solution remains regular as $x\to\infty$ for a semi-infinite transmitting medium. This solution describes wave propagation along the surface, but exponential decay into the medium for $\theta_I > \theta_C$. Such a wave is known as an *evanescent wave*.

The situation with $\theta_I > \theta_C$ is called *total internal reflection* since all the energy in the incident wave is reflected in this case. To see this, we return to Eqs (11.29) and (11.39) for the ratio E_R/E_I. These expressions still hold in the case $\theta_I > \theta_C$, but we should make the replacement

$$\omega n_T\cos\theta_T/c = k_{T,x} = \pm i\kappa\,. \qquad (11.50)$$

It is straightforward then to show that $|E_R|^2 = |E_I|^2$ so that the time-averaged energy flux of the incident wave equals that of the reflected wave. One can also show that the evanescent wave inside the medium only transports energy parallel to the surface on average; there is no flow into the medium.

Total internal reflection has many physical applications: it dictates the geometry in which we should cut diamonds to maximise their "sparkle", and is the reason that we can channel electromagnetic signals in a flexible way through optical fibres.

12 Dispersion

In the discussion above, we assumed that ϵ_r and μ_r were constants. However, in the presence of time-dependent fields, the amplitude and phase of the induced polarisation $\mathbf{P}$ and magnetisation $\mathbf{M}$ will generally depend on the frequency. This gives rise to frequency-dependent $\epsilon_r(\omega)$ and $\mu_r(\omega)$, a phenomenon known as *dispersion*. In this section, we shall see how this frequency dependence can arise with our simple toy model for atomic polarisability, and then explore some of the consequences of this frequency dependence on wave propagation.

12.1 Atomic polarisability revisited: a simple model for $\epsilon_r(\omega)$

Earlier, we considered a simple model for atomic polarisability by thinking about how a rigid electron cloud displaced relative to the nucleus in the presence of a constant applied electric field. If the cloud has charge $-q$, and is displaced by $\mathbf{d}$ relative to the nucleus, the restoring force on it is

$$\mathbf{F}_{\text{restore}} = -\frac{q^2}{4\pi\epsilon_0 a^3}\mathbf{d} = -m\omega_0^2\mathbf{d}\,. \tag{12.1}$$

In the second equality, we have written this restoring force in terms of the reduced mass of the nucleus-cloud system (very nearly the mass of the cloud since electrons are much lighter than neutrons and protons) and the natural oscillation frequency ω_0.

Now that we are considering the time-dependent response of the atom, we should include a phenomenological damping force

$$\mathbf{F}_{\text{damping}} = -\gamma m\dot{\mathbf{d}}\,, \tag{12.2}$$

where γ is some constant with dimensions of frequency. This term is included to allow the oscillating atoms to absorb energy, for example to mimic quantum excitation of the electrons in the atom. If we now apply a time-dependent electric field $\mathbf{E}(t)$ to the atom, the equation of motion for $\mathbf{d}$ is

$$\ddot{\mathbf{d}} = -\frac{q}{m}\mathbf{E}(t) - \omega_0^2\mathbf{d} - \gamma\dot{\mathbf{d}}\,. \tag{12.3}$$

Here, we have assumed that the applied electric field varies slowly in space compared to the atomic scale, in which case the electric field $\mathbf{E}(t)$ can be evaluated at some fixed location within the atom. In this case, Eq. (12.3) becomes the equation of motion of a damped harmonic oscillator.

We consider an applied field of the form $\mathbf{E}(t) = \mathbf{E}_0 e^{-i\omega t}$ (again, it is left implicit that we take the real part), and look for solutions of the form $\mathbf{d}(t) = \mathbf{d}_0 e^{-i\omega t}$. This steady-state solution is

$$\mathbf{d}_0 = -\frac{q\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\,, \tag{12.4}$$

so that the atomic polarisability $\mathbf{p} = \alpha\mathbf{E}$, with $\mathbf{p} = -q\mathbf{d}$, is

$$\alpha = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}\,. \tag{12.5}$$

Note that this is frequency dependent and, generally, complex. The latter means that the electric dipole moment need not oscillate in phase with the applied electric field. As $\omega \to 0$, we recover the earlier result (9.2) for the polarizability in static fields noting that $m\omega_0^2 = q^2/(4\pi\epsilon_0 a^3)$.

If the atoms have number density n, the macroscopic polarization is approximately $\mathbf{P} = n\mathbf{p} = n\alpha\mathbf{E}$. Since $\mathbf{P} = \epsilon_0(\epsilon_r - 1)\mathbf{E}$, we have a frequency-dependent, complex relative permittivity

$$\epsilon_r(\omega) = 1 + \frac{nq^2}{m\epsilon_0(\omega_0^2 - \omega^2 - i\gamma\omega)}\,. \tag{12.6}$$

Although this classical toy model for atomic polarizability is clearly over-simplified, it does capture the basic physics of dispersion.

Normal dispersion and anomalous dispersion

It is generally the case that $\gamma \ll \omega_0$, so that the dimensionless quality factor of the oscillator $Q \equiv \omega_0/\gamma$ is large. In this case, the imaginary part of $\epsilon_r(\omega)$ is small compared to the real part except for frequencies in the vicinity of ω_0. For $\omega < \omega_0$, the real part of $\epsilon_r(\omega)$ is greater than unity, while for $\omega > \omega_0$ it is less than one. At the resonant frequency ω_0, $\epsilon_r(\omega)$ is pure imaginary. This behaviour is illustrated in Fig. 3.

Except in the vicinity of ω_0, the real part of $\epsilon_r(\omega)$ is an increasing function of frequency. Such behaviour is known as *normal dispersion*. Near the resonance, where the imaginary part of $\epsilon_r(\omega)$ becomes appreciable, the real part decreases with frequency; this is referred to as *anomalous dispersion*. We shall see in the next section that for such frequencies there is significant attenuation of waves as the electromagnetic energy in the wave is dissipated in the medium. This arises since the bound current $\partial\mathbf{P}/\partial t$ is almost in phase with the electric field near the resonance.

12.2 Electromagnetic waves in dispersive media

We now explore some of the key differences between propagation of electromagnetic waves in dispersive and non-dispersive media.

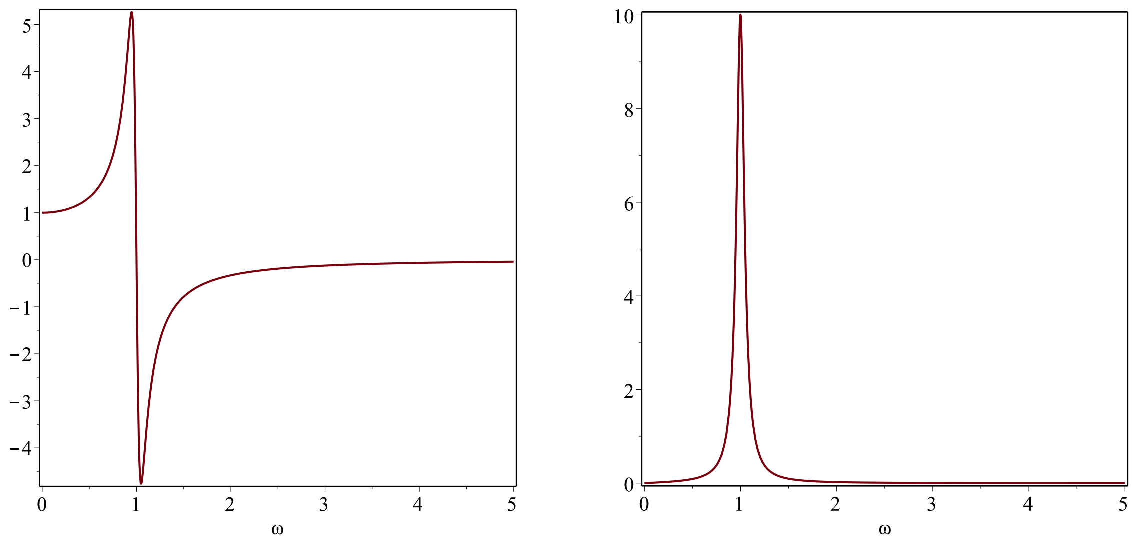

Figure 3: Real part of $\epsilon_r(\omega) - 1$ (left) and imaginary part (right) for $\omega_0 = 1$, $nq^2/(m\epsilon_0\omega_0^2) = 1$ and quality factor $Q = \omega_0/\gamma = 10$.

We look for plane-wave solutions of Maxwell's equations at (real) frequency ω. It is generally a good approximation to take $\mu_r = 1$, but we shall allow for a frequency-dependent $\epsilon_r(\omega)$. Specifically, we have

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}_0(\omega) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{12.7}$$

$$\mathbf{D}(t, \mathbf{x}) = \epsilon_0 \epsilon_r(\omega) \mathbf{E}_0(\omega) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} , \tag{12.8}$$

and for the magnetic field

$$\mathbf{B}(t, \mathbf{x}) = \mathbf{B}_0(\omega) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{12.9}$$

$$\mathbf{H}(t, \mathbf{x}) = \frac{1}{\mu_0} \mathbf{B}_0(\omega) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} . \tag{12.10}$$

We assume there are no free charges or currents present, so two of Maxwell's equations are

$$\nabla \cdot \mathbf{D} = 0 \qquad \Rightarrow \qquad \epsilon_r(\omega) \mathbf{k} \cdot \mathbf{E}_0(\omega) = 0 \tag{12.11}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \qquad \mathbf{k} \cdot \mathbf{B}_0(\omega) = 0 . \tag{12.12}$$

Provided that[17] $\epsilon_r(\omega) \neq 0$, these equations imply that the electric and magnetic fields are transverse to the wavevector $\mathbf{k}$. The other Maxwell equations are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \Rightarrow \qquad \mathbf{k} \times \mathbf{E}_0(\omega) = \omega \mathbf{B}_0(\omega) \tag{12.13}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \qquad \Rightarrow \qquad \mathbf{k} \times \mathbf{B}_0(\omega) = -\frac{\epsilon_r(\omega)}{c^2} \omega \mathbf{E}_0(\omega) . \tag{12.14}$$

[17] We shall meet an example where this condition does not hold in Sec. 13, where we consider wave propagation in conductors.

Taking the cross-product of the latter equation with $\mathbf{k}$, and using $\mathbf{k}\cdot\mathbf{B}_0(\omega) = 0$, we find the *dispersion relation*

$$c^2\mathbf{k}^2 = \epsilon_r(\omega)\omega^2\,. \tag{12.15}$$

As $\epsilon_r(\omega)$ is generally complex, the wavevector will be also.

We already met the idea of a wavevector with complex components in our discussion of total internal reflection in Sec. 11.2.4, where we saw that it corresponds to a decay of the amplitude of the wave in space. (Note, however, that there $\mathbf{k}^2$ was still real.) If we consider a wave propagating in the z-direction, so that $\mathbf{k} = k\hat{\mathbf{z}}$ for complex k, we have

$$c(k_1 + ik_2) = (\epsilon_{r,1} + i\epsilon_{r,2})^{1/2}\omega\,, \tag{12.16}$$

where k_1 and k_2 are the real and imaginary parts, respectively, of k, and similarly for $\epsilon_{r,1}$ and $\epsilon_{r,2}$. The resulting wave is given by the real part of

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}_0(\omega)e^{-k_2 z}e^{i(k_1 z-\omega t)}\,. \tag{12.17}$$

This is decaying along the positive z-direction if $k_2 > 0$.

Finally, we note a further consequence of an imaginary component of $\epsilon_r(\omega)$: we see from $\omega\mathbf{B}_0(\omega) = \mathbf{k}\times\mathbf{E}_0(\omega)$ that there is a phase shift between the oscillations of the electric and magnetic fields for complex $\mathbf{k}$.

12.2.1 Phase and group velocity

We have seen that for a wave of frequency ω, a point of constant phase, such as a zero of the oscillations, has $k_1 z - \omega t = \text{const.}$, where k_1 is the real part of the wavevector. The speed of such a point of constant phase defines the *phase velocity*,

$$v_{\rm p} = \omega/k_1\,. \tag{12.18}$$

In a dispersive medium, $v_{\rm p}$ will generally depend on frequency. This means that refraction will also be frequency dependent, allowing a prism to split white light into its constituent colours. A further consequence of a frequency-dependent $v_{\rm p}$ comes from considering localised wave pulses or packets. These can be formed by superposing waves with a range of frequencies. Since the individual frequency components propagate at different (phase) velocities, the shape of the pulse will typically change as it propagates (see below).

An important property of a wavepacket is the speed at which the peak of the envelope, where the energy is concentrated, moves. A typical situation is to superpose waves with a range of frequencies centred around some central frequency. To simplify our discussion a little, we shall ignore the imaginary part of $\epsilon_r(\omega)$ so that the wavevector is real. If we invert the dispersion relation to view ω as a function of k, we can consider a wavepacket of the form

$$\mathbf{E}(t,z) = \Re \int \frac{dk}{\sqrt{2\pi}}\, \mathbf{E}_0(k) e^{i(kz-\omega(k)t)}\,, \tag{12.19}$$

where $\mathbf{E}_0(k)$ is sharply peaked about some central wavenumber k_0. As a concrete example, consider $\mathbf{E}_0(k) \propto \exp[-(k-k_0)^2/(2\sigma^2)]$. At $t = 0$, the wavepacket is simply the real part of the Fourier transform of $\mathbf{E}_0(k)$, and so

$$\mathbf{E}(t=0,z) \propto \cos(k_0 z) e^{-z^2\sigma^2/2}\,. \tag{12.20}$$

For the typical case of $\sigma \ll k_0$, this is a wide Gaussian modulation of a rapid oscillation at wavenumber k_0. Now consider what happens at later times. For a sharply-peaked $\mathbf{E}_0(k)$, we can expand the dispersion relation $\omega(k)$ about k_0, so the exponent

$$\begin{aligned} kz - \omega(k)t &= kz - \omega(k_0)t - (k-k_0) \left.\frac{d\omega}{dk}\right|_{k_0} + \cdots \\ &= [k_0 z - \omega(k_0)t] + (k-k_0)[z - v_{\mathrm{g}}(k_0)t] + \cdots\,, \end{aligned} \tag{12.21}$$

where we have introduced the *group velocity*

$$v_{\mathrm{g}} = d\omega/dk\,. \tag{12.22}$$

If we retain just these first two terms, the field at time t is

$$\mathbf{E}(t,z) \propto \cos[k_0(z - v_{\mathrm{p}}(k_0)t)] e^{-[z-v_{\mathrm{g}}(k_0)t]^2\sigma^2/2}\,. \tag{12.23}$$

We see that the oscillation propagates at the phase velocity $v_{\mathrm{p}}(k_0)$, while the modulation propagates at the group velocity $v_{\mathrm{g}}(k_0)$. If the medium is non-dispersive, these two velocities are equal (and independent of k_0). Moreover, in this case the expansion in Eq. (12.21) truncates at the order shown, and the wavepacket simply translates along at speed $v_{\mathrm{p}} = v_{\mathrm{g}}$ without changing its shape. In contrast, for a dispersive medium $v_{\mathrm{p}} \neq v_{\mathrm{g}}$ and the oscillation will propagate at a different speed to the wavepacket modulation. Furthermore, inclusion of higher-order terms in the expansion in Eq. (12.21) will generally lead to a change in shape of the modulation as the wavepacket propagates (see Examples 3).

In a dispersive medium, the refractive index is frequency dependent,

$$n(\omega) = c/v_{\mathrm{p}}(\omega)\,. \tag{12.24}$$

We can use this to relate the group and phase velocities (again ignoring any imaginary part of the wavenumber):

$$\frac{1}{v_{\mathrm{g}}} = \frac{dk}{d\omega} = \frac{d}{d\omega}\left(\frac{n\omega}{c}\right) = \frac{1}{v_{\mathrm{p}}} + \frac{\omega}{c}\frac{dn}{d\omega}\,. \tag{12.25}$$

In regions of normal dispersion, $dn/d\omega > 0$ and $v_{\rm g} < v_{\rm p}$; for anomalous dispersion, $dn/d\omega < 0$ and $v_{\rm g} > v_{\rm p}$.

In some situations, the phase velocity can exceed the speed of light in vacuum. An example of this occurs for the simple model of Eq. (12.6) for $\omega \gg \omega_0$. In this limit,

$$\epsilon_r(\omega) \approx 1 - \frac{nq^2}{m\epsilon_0\omega^2}\,. \tag{12.26}$$

Assuming that $\omega^2 > nq^2/(m\epsilon_0)$, the wavevector is approximately real with

$$c^2k^2 = \omega^2 - \frac{nq^2}{m\epsilon_0}\,. \tag{12.27}$$

It follows that the phase velocity

$$v_{\rm p}(\omega) = \frac{c}{\sqrt{1 - nq^2/(m\epsilon_0\omega^2)}}\,, \tag{12.28}$$

which is greater than c. This should not cause alarm – a single-frequency wave extends throughout all of space and cannot be used to communicate any information. To do so requires sending a wave pulse, the peak of which will propagate at the group velocity. Here, $v_{\rm g}(\omega)v_{\rm p}(\omega) = c^2$, so the group velocity is indeed less than c.

12.3 Causality and the Kramers–Kronig relations

A further consequence of the frequency dependence of $\epsilon_r(\omega)$ is a non-locality in time of the connection between $\mathbf{D}(t, \mathbf{x})$ and $\mathbf{E}(t, \mathbf{x})$. To see this, consider writing a general $\mathbf{D}(t, \mathbf{x})$ in terms of its Fourier transform in time:

$$\mathbf{D}(t, \mathbf{x}) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, \tilde{\mathbf{D}}(\omega, \mathbf{x})e^{-i\omega t}\,, \tag{12.29}$$

where $\tilde{\mathbf{D}}(\omega, \mathbf{x})$ is the Fourier transform of the real field $\mathbf{D}(t, \mathbf{x})$ and so satisfies $\tilde{\mathbf{D}}^*(\omega, \mathbf{x}) = \tilde{\mathbf{D}}(-\omega, \mathbf{x})$. We have seen that for oscillations at frequency ω, the field $\mathbf{D}$ and $\mathbf{E}$ at each point in space are related by a factor $\epsilon_0\epsilon_r(\omega)$; it follows that

$$\mathbf{D}(t, \mathbf{x}) = \epsilon_0\mathbf{E}(t, \mathbf{x}) + \epsilon_0 \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, [\epsilon_r(\omega) - 1]\tilde{\mathbf{E}}(\omega, \mathbf{x})e^{-i\omega t}\,, \tag{12.30}$$

where $\tilde{\mathbf{E}}(\omega, \mathbf{x})$ is the Fourier transform of $\mathbf{E}(t, \mathbf{x})$. Note that in extending to negative ω, we must have $\epsilon_r^*(\omega) = \epsilon_r(-\omega)$ from reality of $\mathbf{D}$ and $\mathbf{E}$. In writing Eq. (12.30) in terms of $\epsilon_r(\omega) - 1$, we have isolated the contribution to $\mathbf{D}$ from the polarization in the material in the second term on the right-hand side. It is this term that can be non-local in time. If we express $\tilde{\mathbf{E}}(\omega, \mathbf{x})$ as a Fourier integral, we find

$$\begin{aligned}\mathbf{D}(t, \mathbf{x}) &= \epsilon_0 \mathbf{E}(t, \mathbf{x}) + \epsilon_0 \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \left[\epsilon_r(\omega) - 1\right] \int \frac{dt'}{\sqrt{2\pi}} \mathbf{E}(t', \mathbf{x}) e^{-i\omega(t-t')} \\ &= \epsilon_0 \mathbf{E}(t, \mathbf{x}) + \epsilon_0 \int_{-\infty}^{\infty} dt' \, G(t - t') \mathbf{E}(t', \mathbf{x}) \,, \qquad (12.31)\end{aligned}$$

where we have introduced the *response function*,

$$G(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\epsilon_r(\omega) - 1\right] e^{-i\omega t} \,. \qquad (12.32)$$

Note that this is simply the Fourier transform of $\epsilon_r(\omega) - 1$. If $\epsilon_r(\omega)$ is independent of frequency, $G(t) \propto \delta(t)$ and the connection between $\mathbf{D}$ and $\mathbf{E}$ is local in time. However, a frequency dependence of $\epsilon_r(\omega)$ will introduce some non-locality.

Example: response function for the simple model of atomic polarisability. For the model in Sec. 12.1, we have

$$\epsilon_r(\omega) - 1 = \frac{nq^2}{m\epsilon_0(\omega_0^2 - \omega^2 - i\gamma\omega)} \,. \qquad (12.33)$$

Considered as a function in the complex plane, this has simple poles at $\omega_\pm$ where

$$\omega_\pm = \pm(\omega_0^2 - \gamma^2/4)^{1/2} - i\gamma/2 \,. \qquad (12.34)$$

Note that these are both in the *lower* half plane. It follows that

$$G(t) = -\frac{nq^2}{m\epsilon_0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{(\omega - \omega_+)(\omega - \omega_-)} \,. \qquad (12.35)$$

We can evaluate this integral by contour integration. For $t < 0$, we can close the contour in the upper half-plane since $e^{-i\omega t}$ then tends to zero exponentially fast with radius R along the semi-circle $|\omega| = R$. As there are no poles in the upper half-plane, we see that $G(t) = 0$ for $t < 0$. For $t > 0$, we must close the contour in the lower half-plane, in which case we pick up a contribution to the integral from the two poles. Summing the residues, we find

$$G(t) = \frac{nq^2}{m\epsilon_0\sqrt{\omega_0^2 - \gamma^2/4}} e^{-\gamma t/2} \sin\left(\sqrt{\omega_0^2 - \gamma^2/4}\, t\right) \Theta(t) \,, \qquad (12.36)$$

where $\Theta(t)$ is the Heaviside function that ensures $G(t) = 0$ for $t < 0$. This is an oscillation at the natural (damped) frequency of the oscillator with a decay of characteristic time $1/\gamma$. The decay time is typically much longer than the oscillation period since $\gamma \ll \omega_0$. This means that the non-locality of the response between $\mathbf{D}$ and $\mathbf{E}$ is confined to time differences of the order of $1/\gamma$. In a physical (i.e., quantum) model, γ is related to the frequency width of the spectral lines emitted by the atom in an excited state, and $1/\gamma \sim 10^{-8}\,$s.

In the above example, the response function vanishes for $t < 0$. This is essential for a causal response: the polarisation of the medium at time t cannot depend on the electric field at later times. If we consider evaluating Eq. (12.32) by contour integration (as we did in the example above), for $t < 0$ we can close the contour in the upper half-plane. If the response function is to vanish, so must the contour integral and hence the contour must not enclose any poles of the integrand. We see that, considered as a function in the complex plane, $\epsilon_r(\omega)$ *must be analytic in the upper half-plane* for the response function to be causal.

12.3.1 Kramers–Kronig relations

The fact that $\epsilon_r(\omega)$ is analytic in the upper half-plane can be used to relate the real and imaginary parts along the real axis. Consider the following principal-value integral along the real axis:

$$I(\omega) = \mathcal{P} \int_{-\infty}^{\infty} \frac{[\epsilon_r(\omega') - 1]}{\omega' - \omega}\, d\omega' \,. \tag{12.37}$$

Taking the principal value[18] is necessary since the integrand has a pole at $\omega' = \omega$. We can evaluate $I(\omega)$ by contour integration by considering a closed contour that runs along the real axis from $\omega' = -\infty$, makes a diversion around the pole on the axis at $\omega' = \omega$ by following clockwise a semi-circle of radius δ (we shall ultimately consider the limit $\delta \to 0$) centred on $\omega' = \omega$, and is completed by a semi-circle at infinity in the upper half-plane. Since the integrand has no poles within this contour, the total integral is zero. Assuming that $\epsilon_r(\omega') - 1$ vanishes as $|\omega'| \to \infty$, there is no contribution from the large semi-circular path in the upper half-plane. It follows that $I(\omega) + I_1(\omega) = 0$, where $I_1(\omega)$ is the contribution from the small semi-circular contour about $\omega' = \omega$. This can be evaluated by writing $\omega' - \omega = \delta e^{i\varphi}$, so that

$$\begin{aligned} I_1(\omega) &= \lim_{\delta \to 0} \int_{\pi}^{0} \frac{[\epsilon_r(\omega + \delta e^{i\varphi}) - 1]}{\delta e^{i\varphi}}\, i\delta e^{i\varphi} d\varphi \\ &= -i\pi[\epsilon_r(\omega) - 1]\,, \end{aligned} \tag{12.38}$$

which gives

$$\epsilon_r(\omega) = 1 + \frac{1}{i\pi}\mathcal{P} \int_{-\infty}^{\infty} \frac{[\epsilon_r(\omega') - 1]}{\omega' - \omega}\, d\omega' \,. \tag{12.39}$$

[18] Recall that the principal value is the limit

$$\mathcal{P} \int_{-\infty}^{\infty} f(\omega')\, d\omega' = \lim_{\delta \to 0} \left(\int_{-\infty}^{\omega-\delta} f(\omega')\, d\omega' + \int_{\omega+\delta}^{\infty} f(\omega')\, d\omega' \right),$$

when the integrand $f(\omega')$ has a single pole at $\omega' = \omega$.

Finally, taking the real and imaginary parts gives

$$\Re\epsilon_r(\omega) = 1 + \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\Im\epsilon_r(\omega')}{\omega' - \omega}\, d\omega' \tag{12.40}$$

$$\Im\epsilon_r(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{[\Re\epsilon_r(\omega') - 1]}{\omega' - \omega}\, d\omega' \,. \tag{12.41}$$

These are the *Kramers–Kronig* relations. They are of very general validity, following from causality of the response function alone. Note how they provide a link between the real and imaginary parts of $\epsilon_r(\omega)$. This is of practical interest since, for example, if one can measure the imaginary part of $\epsilon_r(\omega)$ by studying absorption as a function of frequency (recall that the imaginary part of $\epsilon_r(\omega)$ is associated with absorption of radiation by the material since it describes oscillations of the macroscopic polarisation that are in phase quadrature with the applied electric field), one can infer the real part of $\epsilon_r(\omega)$.

The Kramers–Kronig relations are sometimes written in an alternative form that involves only positive frequencies. Using the symmetry $\epsilon_r^*(\omega) = \epsilon_r(-\omega)$ for real ω, we deduce that along the real axis $\Re\epsilon_r(\omega)$ is an even function and $\Im\epsilon_r(\omega)$ is odd. With this property, Eqs (12.40) and (12.41) can be rewritten as

$$\Re\epsilon_r(\omega) = 1 + \frac{2}{\pi}\mathcal{P}\int_{0}^{\infty} \frac{\omega'\Im\epsilon_r(\omega')}{\omega'^2 - \omega^2}\, d\omega' \tag{12.42}$$

$$\Im\epsilon_r(\omega) = -\frac{2\omega}{\pi}\mathcal{P}\int_{0}^{\infty} \frac{[\Re\epsilon_r(\omega') - 1]}{\omega'^2 - \omega^2}\, d\omega' \,. \tag{12.43}$$

13 Electromagnetic waves in conductors

So far, we have discussed the propagation of electromagnetic waves in insulating materials. We now consider the case of conductors where there can be free currents (and charges) due to the mobile electrons. We begin in Sec. 13.1 with a simple dynamical model – the Drude model – for the conductivity of a metal, before discussing in Sec. 13.2 the consequences of this model for wave propagation in conductors.

13.1 Drude model

In a metal, the outermost electrons of the constituent atoms form a delocalised "sea" of mobile electrons surrounding a lattice of ions (nuclei plus tightly-bound inner electrons). These electrons move in all directions and have a range of energies up to what is called the Fermi energy (typically a few eV), since the Pauli exclusion principle forbids them from all occupying the same quantum state. The Fermi energy is much larger than the typical energy of thermal excitations, $O(k_B T) \sim 25\,\text{meV}$ at temperature $T = 300\,\text{K}$. A given electron has its velocity randomised by "collisions". A proper quantum treatment shows that such collisions occur due to anything that destroys the perfect periodicity of the ionic lattice; this includes thermal vibrations of the lattice, defects or impurities, and even the finite extent of the sample. Here, we denote the mean time between collisions (often simply called the *scattering time*) by τ. For reference, in a metal like copper $\tau \sim 10^{-14}\,\text{s}$. The mean distance that an electron moves between collisions is called the *mean-free path*. Taking the speed that corresponds to the Fermi energy, this gives typical velocities of $10^6\,\text{m}\,\text{s}^{-1}$, and so mean-free paths of $O(10^{-8})\,\text{m}$.

In the absence of an applied field, the *average* electron velocity at a point (i.e., averaged over all electrons present at that point) is zero and so is the free current. However, if we apply an electric field, the mobile electrons will be accelerated by the field and there will be a *collective* change in their velocity with time. This gives a non-zero average electron velocity and hence free current. The dynamics of the average electron velocity arises from the competition between acceleration from the applied field and collisions that tend to return the average velocity to zero. The *Drude model*[19] is a classical description of the non-relativistic average electron velocity $\mathbf{v}$. In a uniform electric field $\mathbf{E}(t)$, it models the dynamics of $\mathbf{v}$ with

$$m_e \frac{d\mathbf{v}}{dt} = -e\mathbf{E} - m\frac{\mathbf{v}}{\tau}\,, \tag{13.1}$$

where m_e is the electron mass and $-e$ is the charge. The first force on the right-hand side is the Lorentz force from the electric field (we can generally ignore the magnetic field). The second term on the right of Eq. (13.1) is a phenomenological drag force that describes the effect of collisions.

Equation (13.1) assumes a uniform field, but is actually applicable also for spatially-varying fields that vary slowly compared to the electron mean-free path. This is because at any point in space, the electrons there have typically travelled one mean-free path since their last collision (at which the direction of their velocity was randomised). The average electron velocity at a point is therefore determined by the integrated effect of the electric field at earlier times of order τ, and throughout a region of space with size of the order of the mean-free path.

Consider applying an oscillating electric field $\mathbf{E}(t) = \Re[\mathbf{E}(\omega)e^{-i\omega t}]$ with wavelength large compared to the electron mean-free path. We look for a solution of Eq. (13.1) for the average velocity of the form $\mathbf{v}(t) = \Re[\mathbf{v}(\omega)e^{-i\omega t}]$, which requires

$$\mathbf{v}(\omega)\left(\frac{1}{\tau} - i\omega\right) = -\frac{e}{m_e}\mathbf{E}(\omega)\,. \tag{13.2}$$

[19]The model was proposed in 1900 by Drude. This was before the invention of quantum theory so he (wrongly!) envisaged collisions as arising from the mobile electrons Coulomb scattering off static charged ions.

The macroscopic free current at any point is

$$\mathbf{J}_{\text{free}}(t,\mathbf{x}) = -en_e\mathbf{v}(t,\mathbf{x})\,, \tag{13.3}$$

where n_e is the number density of the mobile electrons. It follows that the current also oscillates as $\mathbf{J} = \Re[\mathbf{J}(\omega)e^{-i\omega t}]$, with

$$\mathbf{J}(\omega) = \frac{n_e e^2\tau}{m_e}\frac{1}{1-i\omega\tau}\mathbf{E}(\omega)\,. \tag{13.4}$$

This is just Ohm's law, but with a frequency-dependent *optical conductivity*

$$\sigma(\omega) = \frac{\sigma_{\text{DC}}}{1-i\omega\tau}\,, \tag{13.5}$$

where the DC conductivity is

$$\sigma_{\text{DC}} = \frac{n_e e^2\tau}{m_e}\,. \tag{13.6}$$

Only when the frequency is small compared to the scattering rate, i.e., $\omega\tau \ll 1$, do we recover the DC conductivity.

Note that the optical conductivity is generally complex, with

$$\Re\sigma(\omega) = \frac{\sigma_{\text{DC}}}{1+\omega^2\tau^2}\,, \qquad \text{and} \qquad \Im\sigma(\omega) = \frac{\sigma_{\text{DC}}\omega\tau}{1+\omega^2\tau^2}\,. \tag{13.7}$$

Since $\mathbf{J}(\omega) = \sigma(\omega)\mathbf{E}(\omega)$, the real part of the optical conductivity generates current in phase with the applied field. This implies energy dissipation, with work being done on the mobile electrons by the oscillating electric field. However, this does not increase the energy of the electrons; rather the energy is transferred to vibrations of the ions through collisions. For $\omega\tau \gg 1$, the optical conductivity is purely imaginary and independent of the scattering time τ (recall that $\sigma_{\text{DC}} \propto \tau$). Here, the oscillations are so rapid that collisions are irrelevant and the mobile electrons respond to the electric field as if they were completely free electrons.

13.2 Propagation of waves in conductors

We now consider the propagation of electromagnetic waves in conductors. We account for both the dielectric response of the lattice of ions, through a frequency-dependent permittivity $\epsilon_r(\omega)$, and the macroscopic free current and charge density from the mobile electrons. We ignore any magnetic response (so take $\mu_r = 1$).

We look for plane-wave solutions of the familiar form

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}_0(\omega)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{13.8}$$

$$\mathbf{B}(t,\mathbf{x}) = \mathbf{B}_0(\omega)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,. \tag{13.9}$$

The electric displacement is

$$\mathbf{D}(t, \mathbf{x}) = \epsilon_0 \epsilon_r(\omega) \mathbf{E}_0(\omega) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,, \tag{13.10}$$

and the free current

$$\mathbf{J}_{\text{free}}(t, \mathbf{x}) = \sigma(\omega) \mathbf{E}_0(\omega) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,. \tag{13.11}$$

The continuity equation for the free current, $\partial \rho_{\text{free}}/\partial t + \boldsymbol{\nabla} \cdot \mathbf{J}_{\text{free}} = 0$, generally implies that there will be non-zero free charge density with

$$\rho_{\text{free}}(t, \mathbf{x}) = \frac{\sigma(\omega)}{\omega} \mathbf{k} \cdot \mathbf{E}_0(\omega) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,. \tag{13.12}$$

Note that ρ_{free} vanishes if the electric field is transverse, $\mathbf{k} \cdot \mathbf{E}_0(\omega) = 0$.

The Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_{\text{free}}$ gives

$$\left(\epsilon_0 \epsilon_r(\omega) + i \frac{\sigma(\omega)}{\omega} \right) \mathbf{k} \cdot \mathbf{E}_0(\omega) = 0\,, \tag{13.13}$$

and $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ gives

$$\mathbf{k} \cdot \mathbf{B}_0(\omega) = 0\,. \tag{13.14}$$

We see that the magnetic field is always transverse and, generally, the electric field is also. The exception is if $\epsilon_0 \epsilon_r(\omega) + i\sigma(\omega)/\omega = 0$ for some frequency ω; we shall discuss this case separately in Sec. 13.2.3.

The remaining Maxwell equations are $\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\text{free}} + \partial \mathbf{D}/\partial t$, which implies that

$$\mathbf{k} \times \mathbf{B}_0(\omega) = -\frac{\omega}{c^2} \left(\epsilon_r(\omega) + i \frac{\sigma(\omega)}{\epsilon_0 \omega} \right) \mathbf{E}_0(\omega)\,, \tag{13.15}$$

and $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, which gives

$$\mathbf{k} \times \mathbf{E}_0(\omega) = \omega \mathbf{B}_0(\omega)\,. \tag{13.16}$$

These equations have exactly the same form as those we met in Sec. 12.2, when discussing electromagnetic waves in dispersive media, provided that we replace $\epsilon_r(\omega)$ with an effective permittivity:

$$\epsilon_r(\omega) \to \epsilon_r^{\text{eff}}(\omega) = \epsilon_r(\omega) + i \frac{\sigma(\omega)}{\epsilon_0 \omega}\,. \tag{13.17}$$

Here, the first term in $\epsilon_r^{\text{eff}}(\omega)$ arises from the dielectric properties of the ions, while the second term arises from the free currents. We can use all of our previous results; in particular, the dispersion relation is

$$c^2 \mathbf{k}^2 = \epsilon_r^{\text{eff}}(\omega) \omega^2\,. \tag{13.18}$$

We now explore the consequences of this dispersion relation taking the conductivity to be described by the Drude form (13.5).

13.2.1 Low-frequency behaviour

If we write $\epsilon_r(\omega)$ in terms of its real and imaginary parts, $\epsilon_r(\omega) = \epsilon_{r,1}(\omega) + i\epsilon_{r,2}(\omega)$, we have

$$\epsilon_r^{\text{eff}}(\omega) = \epsilon_{r,1}(\omega) + i\epsilon_{r,2}(\omega) + \frac{i}{\epsilon_0\omega}\frac{\sigma_{\text{DC}}}{1 - i\omega\tau} \,. \tag{13.19}$$

For $\omega\tau \ll 1$, the optical conductivity $\sigma(\omega) \approx \sigma_{\text{DC}}$. Moreover, for the simple model of $\epsilon_r(\omega)$ in Eq. (12.6), $\epsilon_{r,1} \gg \epsilon_{r,2}$ except near the resonance at ω_0, and $\epsilon_{r,1}$ tends to a constant as $\omega \to 0$. It follows that at sufficiently low frequencies,

$$\epsilon_r^{\text{eff}}(\omega) \approx i\frac{\sigma_{\text{DC}}}{\epsilon_0\omega} \,. \tag{13.20}$$

At such low frequencies, it is the mobile electrons that totally dominate the electromagnetic properties of the conductor.

Taking the wavevector $\mathbf{k} = k\hat{\mathbf{z}}$ for complex $k = k_1 + ik_2$, we have, from Eq. (13.18),

$$k_1 + ik_2 = \pm\sqrt{\frac{\mu_0\omega\sigma_{\text{DC}}}{2}}(1 + i) \,. \tag{13.21}$$

For propagation along the positive z-direction we should select the positive root, in which case

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}_0(\omega)e^{-z/\delta}e^{i(z/\delta-\omega t)} \,, \tag{13.22}$$

where the *skin-depth*

$$\delta \equiv \sqrt{\frac{2}{\mu_0\omega\sigma_{\text{DC}}}} \,. \tag{13.23}$$

We see that we have a propagating wave, but with amplitude that decays along the propagation direction over a characteristic length given by the skin depth. Note that δ decreases with increasing frequency so that higher frequency electromagnetic fields are more confined to the surfaces of conductors. The amplitude of the wave decays as energy is dissipated by the free current, which oscillates in phase with the electric field.

We can find the magnetic field from $\mathbf{k} \times \mathbf{E}_0(\omega) = \omega\mathbf{B}_0(\omega)$, so that

$$\mathbf{B}_0(\omega) = e^{i\pi/4}\sqrt{\frac{\mu_0\sigma_{\text{DC}}}{\omega}}\hat{\mathbf{z}} \times \mathbf{E}_0(\omega) \,. \tag{13.24}$$

The magnetic field is $\pi/4$ out of phase with the electric field, and the amplitude of the magnetic field can be much larger than the electric field in good conductors at low frequency.

13.2.2 High-frequency behaviour

We now consider high frequencies, $\omega\tau \gg 1$. In this limit, collisions play no role and $\sigma(\omega) \approx i\sigma_{\rm DC}/(\omega\tau)$, which is independent of τ. We shall also assume that $\omega \gg \omega_0$ so that $\epsilon_r(\omega) \approx 1$. It follows that

$$\begin{aligned} \epsilon_r^{\rm eff} &\approx 1 - \frac{\sigma_{\rm DC}}{\epsilon_0 \omega^2 \tau} \\ &= 1 - \frac{\omega_p^2}{\omega^2}\,, \end{aligned} \tag{13.25}$$

where we have introduced the *plasma frequency*

$$\omega_p^2 \equiv \frac{n_e e^2}{\epsilon_0 m_e}\,. \tag{13.26}$$

In a typical metal, $\omega_p \sim 10^{16}\,\mathrm{rad\,s^{-1}}$. There is qualitatively different behaviour for $\omega > \omega_p$ or $\omega < \omega_p$, as follows.

- $\omega > \omega_p$: in this case, $\epsilon_r^{\rm eff}(\omega) > 0$ and the wavenumber k is real. We have simple wave propagation in the conductor with no dissipation (because $\mathbf{J}_{\rm free}$ and $\mathbf{E}$ are out of phase). This shows that metals are partially transparent above their plasma frequency. The waves are dispersive, with dispersion relation

  $$c^2\mathbf{k}^2 = \omega^2 - \omega_p^2\,. \tag{13.27}$$

  The phase velocity is superluminal:

  $$v_{\rm p}(\omega) = \frac{c}{\sqrt{1-\omega_p^2/\omega^2}}\,, \tag{13.28}$$

  but the group velocity is less than the speed of light since $v_{\rm g}(\omega) = c^2/v_{\rm p}(\omega)$.
- $\omega < \omega_p$: in this limit[20], $\epsilon_r^{\rm eff}(\omega) < 0$ and the wavenumber k is purely imaginary. We therefore have an evanescent wave: *wave propagation in a conductor is not possible below the plasma frequency (but above the collision rate).* In this case, all of the energy of an electromagnetic wave that is incident on a conductor will be reflected.

Since the plasma frequency typically corresponds to ultraviolet radiation, optical light has frequency below the plasma frequency and is strongly reflected from conductors. This is why metals are shiny.

[20]Given our assumptions that $\omega\tau \gg 1$ and $\omega \gg \omega_0$, this requires $\omega_p \gg \max(\tau^{-1}, \omega_0)$. The condition on the collision rate is generally met, since $\tau^{-1} \sim 10^{14}\,\mathrm{s^{-1}}$, but for $\hbar\omega_0$ of the order of atomic transition energies the condition on ω_0 is not so obviously met.

13.2.3 Plasma oscillations

We now return to Eq. (13.13), which we can rewrite as

$$\epsilon_0 \epsilon_r^{\text{eff}}(\omega) \mathbf{k} \cdot \mathbf{E}_0(\omega) = 0 \,. \tag{13.29}$$

As noted earlier, if $\epsilon_r^{\text{eff}}(\omega)$ vanishes for some ω, there are solutions with $\mathbf{k} \cdot \mathbf{E}_0(\omega) \neq 0$. Moreover, in this case, Eqs (13.14) and (13.15) give $\mathbf{k} \cdot \mathbf{B}_0(\omega) = 0$ and $\mathbf{k} \times \mathbf{B}_0(\omega) = 0$, which imply that the magnetic field vanishes. Equation (13.16) further tells us that $\mathbf{k} \times \mathbf{E}_0(\omega) = 0$, so we have a new solution that is a *longitudinal wave* with $\mathbf{k}$ parallel to $\mathbf{E}$.

We can find oscillatory solutions of this form by considering the high-frequency limit of $\epsilon_r^{\text{eff}}(\omega)$ in Eq. (13.25). This vanishes if $\omega = \omega_p$, giving solutions

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}_0(\omega_p) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_p t)} \,, \tag{13.30}$$

with $\mathbf{k} \times \mathbf{E}_0(\omega) = 0$. Since these solutions are longitudinal waves, the charge density is non-zero and also oscillates at ω_p:

$$\rho_{\text{free}}(t, \mathbf{x}) = i\epsilon_0 \mathbf{k} \cdot \mathbf{E}_0(\omega_p) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_p t)} \,, \tag{13.31}$$

where we have approximated $\epsilon_r(\omega_p) \approx 1$. Note that the wavevector $\mathbf{k}$ is arbitrary, so, more generally, we have

$$\mathbf{E}(t, \mathbf{x}) = -\boldsymbol{\nabla} \phi(\mathbf{x}) e^{-i\omega_p t} \,, \tag{13.32}$$

for some potential ϕ. The spatial profile of the free charge density is controlled by $\nabla^2 \phi$ and is arbitrary. These oscillations are called *plasma oscillations*.

A Review of special relativity

This appendix recaps the main ideas of special relativity and reviews the streamlined tensor notation that was introduced in IB *Electromagnetism* and is used throughout this course. Note that the metric signature adopted here is mostly positive, i.e., the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

Special relativity is based on the following two postulates:

- The laws of physics are the same in all inertial frames[21].
- Light signals in vacuum propagate rectilinearly and with the same speed c in all inertial frames.

The first postulate means that any equation of physics must take the same form when written in terms of spacetime coordinates in different inertial frames.

A.1 Lorentz transformations

The consequence of these two postulates is that for two inertial frames, S and S', with spatial axes aligned and with S' moving at speed $v = \beta c$ along the positive x-axis of S and with origins coincident at $t = t' = 0$, the spacetime coordinates of any event are related by the (standard) Lorentz transformation:

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \,, \end{aligned} \tag{A.1}$$

where $\gamma \equiv (1 - \beta^2)^{-1/2}$. The transformation for a general relative velocity $\mathbf{v}$, with axes still aligned, can be obtained by first co-rotating the spatial axes of S and S' so that the relative motion is along the x-axis, performing the above transformation, and then back-rotating. Such Lorentz transformations are called *Lorentz boosts* and they

[21]Inertial frames are frames of reference in which a freely-moving particle (i.e., one subject to no external forces) proceeds at constant velocity in accordance with Newton's first law. A frame of reference can be thought of as a set of test particles, all at rest with respect to each other, and each equipped with a synchronised clock. Laying the test particles out on a Cartesian grid and assigning spatial coordinates, spacetime coordinates can be assigned to any event in spacetime by the spatial coordinates of the test particle on whose wordline the event occurs, and by the time on the test particle's clock at the event.

contain no additional relative (spatial) rotation between the two frames. Note that the Lorentz transformation of Eq. (A.1) reduces to a Galilean transformation for $\beta \ll 1$.

By construction, Lorentz transformations preserve the spacetime interval between two events, i.e.,

$$(\Delta \mathbf{x}')^2 - (c\Delta t')^2 = (\Delta \mathbf{x})^2 - (c\Delta t)^2 \,. \tag{A.2}$$

If we write the spacetime coordinates as $x^\mu = (ct, x_i)$, with $\mu = 0$ labelling the time component, and $\mu = i = 1, 2, 3$ labelling the space components, we can write the invariant interval for infinitesimal separations in the form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,. \tag{A.3}$$

Here, $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the *Minkowski metric* and we have introduced the summation convention that *repeated spacetime (Greek) indices, with one upper and one lower, are summed over.*

More generally, we define *homogeneous Lorentz transformations* to be transformations from one system of spacetime coordinates x^μ to another x'^μ, with

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \,, \tag{A.4}$$

such that

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \,. \tag{A.5}$$

Equivalently, we can write this in matrix form as $\boldsymbol{\Lambda}^T \boldsymbol{\eta} \boldsymbol{\Lambda} = \boldsymbol{\eta}$. Such transformations leave $\eta_{\mu\nu} dx^\mu dx^\nu$ invariant since

$$\begin{aligned} \eta_{\mu\nu} dx'^\mu dx'^\nu &= \underbrace{\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta}_{\eta_{\alpha\beta}} dx^\alpha dx^\beta \\ &= \eta_{\alpha\beta} dx^\alpha dx^\beta \,. \end{aligned} \tag{A.6}$$

Note that when we sum over repeated indices, the exact symbol we assign to the "dummy" index is irrelevant. For the standard transformation of Eq. (A.1), we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{A.7}$$

Homogeneous Lorentz transformations form a *group*. Closure in ensured since, for a composition of two transformations represented by the matrix product $\boldsymbol{\Lambda}_1 \boldsymbol{\Lambda}_2$, we have

$$\begin{aligned} (\boldsymbol{\Lambda}_1 \boldsymbol{\Lambda}_2)^T \boldsymbol{\eta} (\boldsymbol{\Lambda}_1 \boldsymbol{\Lambda}_2) &= \boldsymbol{\Lambda}_2^T \boldsymbol{\Lambda}_1^T \boldsymbol{\eta} \boldsymbol{\Lambda}_1 \boldsymbol{\Lambda}_2 \\ &= \boldsymbol{\Lambda}_2^T \boldsymbol{\eta} \boldsymbol{\Lambda}_2 \\ &= \boldsymbol{\eta} \,. \end{aligned} \tag{A.8}$$

Clearly, the set of homogeneous Lorentz transformations contains the identity, $\Lambda^\mu{}_\nu = \delta^\mu_\nu$, and every element is invertible[22] since $\mathbf{\Lambda}^T \boldsymbol{\eta} \mathbf{\Lambda} = \boldsymbol{\eta}$ implies that $\det(\mathbf{\Lambda})^2 = 1$. We shall be most interested in the subgroup of *restricted homogeneous Lorentz transformations*. These are transformations that are continuously connected to the identity, i.e., any element can be taken to the identity by smooth variation of its parameters. In particular, restricted Lorentz transformations have $\det(\mathbf{\Lambda}) = +1$ and[23] $\Lambda^0{}_0 \geq 1$. Physically, they correspond to transformations between inertial frames that exclude space inversion or time reversal.

Since $\eta_{\mu\nu}$ is symmetric, Eq. (A.5) puts 10 independent constraints on the 16 independent components of $\Lambda^\mu{}_\nu$. It follows that a general homogeneous Lorentz transformation is described by $16 - 10 = 6$ parameters. Three of these encode the relative velocity of the two frames (i.e., the boost part) and three any additional spatial rotation.

The invariant interval $\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu x^\nu$ between two events allows us to classify all spacetime events in relation to some given event as being either *spacelike separated* ($\Delta s^2 > 0$), *timelike* ($\Delta s^2 < 0$) or null (lightlike, with $\Delta s^2 = 0$). At any event, the set of null-separated events defines the *light cone*. Light rays emitted from some event and propagating in vacuum have worldlines that lie on the lightcone of the emission event. Massive particles follow worldlines for which the tangent vector is timelike (i.e., has negative norm) and at any event a massive particle will be moving within the lightcone in spacetime. For timelike- and null-separated events, the temporal ordering of the events (i.e., the sign of Δt) is Lorentz invariant so we can speak of the future and past light cone. This is not the case for spacelike-separated events.

A.2 4-vectors

In Newtonian physics, it is very convenient to write our equations in terms of 3D vectors (or tensors) to make manifest their transformation properties under spatial rotations. We now generalise this idea to spacetime 4-vectors and tensors. By expressing our equations in terms of such objects, we can be sure that we are writing down equations that are form invariant under Lorentz transformations and hence satisfy the postulates of special relativity.

[22] The inverse of a Lorentz transformation is also a Lorentz transformation since multiplying $\mathbf{\Lambda}^T \boldsymbol{\eta} \mathbf{\Lambda} = \boldsymbol{\eta}$ on the left with $(\mathbf{\Lambda}^T)^{-1}$ and on the right with $\mathbf{\Lambda}^{-1}$ implies that $(\mathbf{\Lambda}^{-1})^T \boldsymbol{\eta} \mathbf{\Lambda}^{-1} = \boldsymbol{\eta}$. Physically, this makes sense since the inverse transforms back from x'^μ to x^μ.

[23] Note that the 00 element of Eq. (A.5) gives

$$\eta_{00} = \eta_{00}(\Lambda^0{}_0)^2 + \sum_i \eta_{ii}(\Lambda^i{}_0)^2$$

$$\Rightarrow \quad (\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2 \geq 1\,.$$

We define a 4-vector to be an object A^μ that transforms like the displacement dx^μ under Lorentz transformations, i.e.,

$$A^\mu \to \Lambda^\mu{}_\nu A^\nu \quad \text{for } x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \,. \tag{A.9}$$

It follows that $\eta_{\mu\nu}A^\mu A^\nu$ is invariant, defining a Lorentz-invariant norm. More generally, we can define a Lorentz-invariant scalar product between two 4-vectors, A^μ and B^μ as $\eta_{\mu\nu}A^\mu B^\nu$.

We should properly call A^μ a *contravariant* 4-vector to distinguish it from a *covariant* 4-vector. To motivate the introduction of the latter, consider the transformation of the differential operator $\partial/\partial x^\mu$: the chain rule gives

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \,. \tag{A.10}$$

Since

$$\frac{\partial x^\nu}{\partial x'^\mu}\underbrace{\frac{\partial x'^\mu}{\partial x^\rho}}_{\Lambda^\mu{}_\rho} = \delta^\nu_\rho \,, \tag{A.11}$$

it follows that $\partial x^\nu/\partial x'^\mu = (\Lambda^{-1})^\nu{}_\mu$, where the right-hand side is the matrix representing the inverse transformation. The Lorentz transformation law for partial derivatives is therefore

$$\frac{\partial}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu} \,. \tag{A.12}$$

Note the distinction to the transformation law for a contravariant 4-vector. It is convenient to *define* the quantities

$$\Lambda_\mu{}^\nu \equiv (\Lambda^{-1})^\nu{}_\mu \,; \tag{A.13}$$

we then define covariant 4-vectors, A_μ, to be objects that transform under Lorentz transformations as

$$A'_\mu = \Lambda_\mu{}^\nu A_\nu \,. \tag{A.14}$$

Note the consistent index placement on both sides of all equations.

We can express $\Lambda_\mu{}^\nu$ in terms of $\Lambda^\mu{}_\nu$ and the Minkowski metric directly by noting that

$$\boldsymbol{\Lambda}^T \boldsymbol{\eta} \boldsymbol{\Lambda} = \boldsymbol{\eta} \quad \Rightarrow \quad \boldsymbol{\Lambda}^{-1} = \boldsymbol{\eta}^{-1}\boldsymbol{\Lambda}^T\boldsymbol{\eta} \,, \tag{A.15}$$

or, in terms of components,

$$\begin{aligned} \Lambda_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu &= \eta^{\nu\rho}\Lambda^\tau{}_\rho \eta_{\tau\mu} \\ &= \eta_{\mu\tau}\eta^{\nu\rho}\Lambda^\tau{}_\rho \,, \end{aligned} \tag{A.16}$$

where we have written the components of the inverse of the metric as $\eta^{\mu\nu}$. Numerically, the components $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ are the same, and

$$\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu \,. \tag{A.17}$$

The *contraction* of a contravariant vector A_μ and a covariant vector B^μ is Lorentz invariant since

$$\begin{aligned} A'_\mu B'^\mu &= \Lambda_\mu{}^\nu A_\nu \Lambda^\mu{}_\tau B^\tau \\ &= (\Lambda^{-1})^\nu{}_\mu \Lambda^\mu{}_\tau A_\nu B^\tau \\ &= \delta^\nu_\tau A_\nu B^\tau \\ &= A_\nu B^\nu \,. \end{aligned} \tag{A.18}$$

Every contravariant vector can be mapped with the Minkowski metric into an associated covariant vector and vice versa. For example, we can write

$$A_\mu = \eta_{\mu\nu} A^\nu , \quad A^\mu = \eta^{\mu\nu} A_\nu \,. \tag{A.19}$$

Explicitly, this means that $A_0 = -A^0$ and $A_i = A^i$. We can easily check that $\eta_{\mu\nu} A^\nu$ is a covariant vector by considering its transformation law:

$$\begin{aligned} A'_\mu &= \eta_{\mu\nu} A'^\nu \\ &= \eta_{\mu\nu} \Lambda^\nu{}_\tau A^\tau \\ &= \underbrace{\eta_{\mu\nu} \Lambda^\nu{}_\tau \eta^{\tau\rho}}_{\Lambda_\mu{}^\rho} A_\rho \\ &= \Lambda_\mu{}^\rho A_\rho \,. \end{aligned} \tag{A.20}$$

The scalar product between two contravariant vectors, A^μ and B^μ, can then be written either as $\eta_{\mu\nu} A^\mu B^\nu$, or as a contraction $A^\mu B_\mu$ between the contravariant A^μ and the covariant B_μ (or the other way around).

For the covariant differential operator $\partial_\mu \equiv \partial/\partial x^\mu$, we can form the Lorentz-invariant (scalar) operator

$$\Box \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu = -\frac{\partial^2}{\partial (ct)^2} + \nabla^2 \,. \tag{A.21}$$

This is the operator that appears in the wave equation for waves propagating at speed c.

Alternative take on covariant and contravariant components of 4-vectors: In 3D Euclidean space we are used to thinking of vectors as geometric objects, the archetypal example being displacement vectors that connect two points in space. Let us generalise this idea to spacetime so that the 4-vector $d\mathbf{x}$ connects two neighbouring events[24]. By taking unit displacements along the coordinate axes of some inertial frame, we can introduce a basis of four vectors $\mathbf{e}_\mu$. Here, the subscript μ labels the element of the basis. In terms of this basis, we can write

$$d\mathbf{x} = dx^\mu \mathbf{e}_\mu \,. \tag{A.22}$$

[24]We shall temporarily denote 4-vectors by boldface in this brief interlude.

Under a Lorentz transformation (i.e., a passive reparameterisation of spacetime with a different set of inertial coordinates) the coordinate differentials dx^μ transform but the 4-vector itself is unchanged. This means that the basis vectors must also change to compensate the change in the dx^μ. Indeed, since $\mathbf{e}_\mu = \partial \mathbf{x}/\partial x^\mu$, we have

$$\mathbf{e}'_\mu = \frac{\partial \mathbf{x}}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial \mathbf{x}}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\nu}\mathbf{e}_\nu = \Lambda_\mu{}^\nu \mathbf{e}_\nu \, . \tag{A.23}$$

We introduce an inner product on spacetime; this is a bilinear symmetric function of two 4-vectors, $\mathbf{A} = A^\mu \mathbf{e}_\mu$ and $\mathbf{B} = B^\mu \mathbf{e}_\mu$, defined by

$$\mathbf{A}\cdot\mathbf{B} \equiv \eta_{\mu\nu} A^\mu B^\nu \, . \tag{A.24}$$

It follows that we must have

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \eta_{\mu\nu} \, , \tag{A.25}$$

and it is easy to verify that this is preserved under Lorentz transformations.

The *dual basis* of 4-vectors, denoted $\mathbf{e}^\mu$, is defined by

$$\mathbf{e}^\mu \cdot \mathbf{e}_\nu = \delta^\mu_\nu \, , \tag{A.26}$$

so that $\mathbf{e}^\mu = \eta^{\mu\nu}\mathbf{e}_\nu$ and $\mathbf{e}_\mu = \eta_{\mu\nu}\mathbf{e}^\nu$. The dual basis transforms in the same way as the dx^μ under Lorentz transformations, $\mathbf{e}'^\mu = \Lambda^\mu{}_\nu \mathbf{e}^\nu$, to preserve the duality relation in Eq. (A.26). We can express a 4-vector $\mathbf{A}$ in terms of its components on either basis, i.e.,

$$\mathbf{A} = A^\mu \mathbf{e}_\mu = A_\mu \mathbf{e}^\mu \, . \tag{A.27}$$

As usual, we can extract the components by taking the inner product with the appropriate basis 4-vectors: $A^\mu = \mathbf{A}\cdot\mathbf{e}^\mu$ and $A_\mu = \mathbf{A}\cdot\mathbf{e}_\mu$. It follows that

$$A_\mu = (A^\nu \mathbf{e}_\nu)\cdot\mathbf{e}_\mu = \eta_{\mu\nu}A^\nu \, , \tag{A.28}$$

and, similarly, $A^\mu = \eta^{\mu\nu}A_\nu$. Under Lorentz transformations,

$$A'_\mu = \mathbf{A}\cdot\mathbf{e}'_\mu = \mathbf{A}\cdot(\Lambda_\mu{}^\nu \mathbf{e}_\nu) = \Lambda_\mu{}^\nu A_\nu \, , \tag{A.29}$$

so the A_μ (the *covariant* components) transform in the *same* way as the coordinate basis vectors. This is the origin of the covariant and contravariant terminology for the components.

We see that the covariant components A_μ and the contravariant components A^μ are an equivalent description of the same geometric object, the 4-vector $\mathbf{A}$. Some 4-vectors are more naturally represented by their contravariant components, an obvious example being the dx^μ of the spacetime displacement, while for others the contravariant components are more natural. An example of the latter are the $\partial/\partial x^\mu$ of the spacetime gradient operator.

A.2.1 Examples of 4-vectors in relativistic kinematics

Velocity 4-vector. Consider a point particle following a worldline $x^\mu(\tau)$, where τ is the particle's proper time (i.e., time as measured on a clock carried with the particle). The velocity 4-vector is defined by

$$u^\mu = dx^\mu/d\tau \,, \tag{A.30}$$

where we are dividing a 4-vector dx^μ by the scalar $d\tau$. We can express the proper time in terms of the time in any inertial frame S by noting that

$$\begin{aligned} c^2 d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu &= c^2 dt^2 - d\mathbf{x}^2 \\ &= c^2 dt^2 (1 - \mathbf{v}^2/c^2) \\ &= c^2 dt^2/\gamma^2 \,, \end{aligned} \tag{A.31}$$

where γ is the Lorentz factor associated with the particle's velocity $\mathbf{v}$ relative to S. It follows that $d\tau = dt/\gamma$, and so the components of the velocity 4-vector in S are

$$u^\mu = \gamma \frac{d}{dt}(ct, \mathbf{x}) = (\gamma c, \gamma \mathbf{v}) \,. \tag{A.32}$$

Note the norm of u^μ:

$$\eta_{\mu\nu} u^\mu u^\nu = (\eta_{\mu\nu} dx^\mu dx^\nu)/(d\tau)^2 = -c^2 \,. \tag{A.33}$$

Momentum 4-vector. For a massive particle of rest-mass m, the momentum 4-vector is obtained from the velocity 4-vector by multiplication by m:

$$p^\mu \equiv m u^\mu = (\gamma m c, \gamma m \mathbf{v}) \,. \tag{A.34}$$

The norm is $\eta_{\mu\nu} p^\mu p^\nu = -m^2 c^2$. In some inertial frame, the components of p^μ are E/c, the (total) energy measured in that frame, and $\mathbf{p}$, the relativistic 3-momentum, i.e.,

$$p^\mu = (E/c, \mathbf{p}) \quad \text{where } E = \gamma m c^2 \text{ and } \mathbf{p} = \gamma m \mathbf{v} \,. \tag{A.35}$$

For an isolated system, the 4-momentum is *conserved*, i.e., the relativistic 3-momentum and energy are conserved.

More generally, the 3-momentum of a particle is changed by a force $\mathbf{F}$ according to Newton's second law $d\mathbf{p}/dt = \mathbf{F}$. The interpretation of the time component of p^μ as the energy is then consistent since differentiating $\eta_{\mu\nu} p^\mu p^\nu = -m^2 c^2$ gives

$$\begin{aligned} E dE/dt &= c^2 \mathbf{p} \cdot d\mathbf{p}/dt \\ &= c^2 \mathbf{p} \cdot \mathbf{F} \\ \Rightarrow \quad dE/dt &= \mathbf{p} \cdot \mathbf{F}/(\gamma m) = \mathbf{v} \cdot \mathbf{F} \,, \end{aligned} \tag{A.36}$$

where the quantity on the right of the final equality is the usual rate of working of the force. We can introduce a 4-vector force F^μ as $F^\mu = dp^\mu/d\tau$; then

$$\begin{aligned} F^\mu &= \gamma \frac{d}{dt}\left(\frac{E}{c}, \mathbf{p}\right) \\ &= \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma \mathbf{F}\right) \\ &= \left(\frac{\gamma}{c}\mathbf{v}\cdot\mathbf{F}, \gamma\mathbf{F}\right) . \end{aligned} \tag{A.37}$$

The 4-force is necessarily orthogonal to p^μ (and so to u^μ) to preserve the norm of p^μ.

Acceleration 4-vector. The 4-acceleration a^μ is

$$a^\mu = du^\mu/d\tau \, . \tag{A.38}$$

Constancy of the norm $\eta_{\mu\nu}u^\mu u^\nu = -c^2$ implies that a^μ is orthogonal to u^μ. The components of a^μ are

$$\begin{aligned} a^\mu &= \gamma \frac{d}{dt}(\gamma c, \gamma \mathbf{v}) \\ &= \gamma \left(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\mathbf{v} + \gamma \mathbf{a}\right) \\ &= \gamma^2 \left(\frac{\gamma^2}{c}\mathbf{v}\cdot\mathbf{a}, \frac{\gamma^2}{c^2}\mathbf{a}\cdot\mathbf{v}\mathbf{v} + \mathbf{a}\right) , \end{aligned} \tag{A.39}$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the usual 3-acceleration and we used $d\gamma/dt = \gamma^3 \mathbf{a}\cdot\mathbf{v}/c^2$, which follows from differentiating $\gamma^{-2} = 1 - \mathbf{v}\cdot\mathbf{v}/c^2$. In the *instantaneous rest frame* of the particle (i.e., the frame in which it is at rest at any instant), the components of the 4-acceleration are simply $a^\mu = (0, \mathbf{a}_{\text{rest}})$. It follows that the norm of a^μ is the square of the rest-frame acceleration: $\eta_{\mu\nu}a^\mu a^\nu = \mathbf{a}^2_{\text{rest}}$.

Example: A particle moves along the x-axis of some inertial frame S with constant rest-frame acceleration α. At proper time τ, suppose that the particle is at $x(\tau)$ and $t(\tau)$ in S and is moving with speed βc (along the x-axis). In its instantaneous rest frame, the components of the 4-acceleration are $a'^\mu = \alpha(0,1,0,0)$. Performing the inverse Lorentz transform back to S gives

$$a^\mu = du^\mu/d\tau = (\gamma\beta\alpha, \gamma\alpha, 0, 0) \, . \tag{A.40}$$

Since the velocity 4-vector is $u^\mu = (\gamma c, \gamma c \beta, 0, 0)$, we have

$$\frac{d(\gamma c)}{d\tau} = \gamma\beta\alpha \quad \text{and} \quad c\frac{d(\gamma\beta)}{d\tau} = \gamma\alpha \, . \tag{A.41}$$

To solve these, we note that $d\gamma/d\tau = \gamma^3\beta d\beta/d\tau$; combining with Eq. (A.41), we find

$$\frac{d\beta}{d\tau} = \frac{\alpha}{\gamma^2 c} = \frac{\alpha}{c}(1-\beta^2) \, . \tag{A.42}$$

The solution with $\beta = 0$ at $\tau = 0$ is

$$\beta(\tau) = \tanh(\alpha\tau/c) \quad \text{and} \quad \gamma(\tau) = \cosh(\alpha\tau/c)\,. \tag{A.43}$$

As expected, the speed asymptotes to c ($\beta = 1$). We can solve for the wordline of the particle using

$$\frac{dx}{d\tau} = \gamma\frac{dx}{dt} = \gamma c\beta = c\sinh(\alpha\tau/c)\,. \tag{A.44}$$

The solution with $x = c^2\alpha$ at $\tau = 0$ (for convenience) is

$$x(\tau) = \frac{c^2}{\alpha}\cosh(\alpha\tau/c)\,. \tag{A.45}$$

Similarly, $dt/d\tau = \gamma = \cosh(\alpha\tau/c)$ gives

$$t(\tau) = \frac{c}{\alpha}\sinh(\alpha\tau/c)\,, \tag{A.46}$$

taking $t = 0$ at $\tau = 0$. We see that the motion is hyperbolic in the x-t plane, and asymptotes to $x = c|t|$ as $|\tau| \to \infty$. The wordline is entirely within the "Rindler wedge", $x > c|t|$. Note that if we consider an observer at rest in S at the origin, sending out light signals to the accelerated particle, only signals emitted before $t = 0$ are ever received by the accelerated particle.

A.3 4-tensors

A tensor of type $\binom{p}{q}$ has p upper (contravariant) indices and q lower (covariant) indices. Such a tensor $T^{\mu\nu\ldots}{}_{\rho\sigma\ldots}$ transforms as

$$T'^{\mu\nu\ldots}{}_{\rho\sigma\ldots} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\ldots\Lambda_\rho{}^\gamma\Lambda_\sigma{}^\delta\ldots T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} \tag{A.47}$$

under $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. As with 4-vectors, one can raise and lower indices with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, for example

$$T^{\mu\nu} = \eta^{\nu\alpha}T^\mu{}_\alpha \tag{A.48}$$

which takes a type-$\binom{1}{1}$ tensor to a type $\binom{2}{0}$.

The following rules apply to tensor manipulations.

- Contractions may only be made between upper and lower indices. Contracting a $\binom{p}{q}$ tensor returns a $\binom{p-1}{q-1}$ tensor. For example $T^\alpha{}_\alpha$ is a Lorentz scalar since

$$\begin{aligned} T'^\alpha{}_\alpha &= \underbrace{\Lambda^\alpha{}_\rho\Lambda_\alpha{}^\delta}_{\delta^\delta_\rho} T^\rho{}_\delta \\ &= T^\rho{}_\rho\,. \end{aligned} \tag{A.49}$$

Note that the specific symbol assigned to a contracted (dummy) index is irrelevant.

- Symmetrisation/antisymmetrisation may be performed only over upper or lower indices (but not mixed). Symmetrisation is denoted with round brackets and antisymmetrisation with square brackets:

$$T^{(\alpha\beta)} \equiv \frac{1}{2}\left(T^{\alpha\beta} + T^{\beta\alpha}\right) \quad \text{and} \quad T^{[\alpha\beta]} \equiv \frac{1}{2}\left(T^{\alpha\beta} - T^{\beta\alpha}\right) . \tag{A.50}$$

- Contraction, symmetrisation and index raising/lowering commute with Lorentz transformations. For example,

$$\begin{aligned} 2T'^{(\alpha\beta)} &= T'^{\alpha\beta} + T'^{\beta\alpha} \\ &= \Lambda^{\alpha}{}_{\sigma}\Lambda^{\beta}{}_{\rho}T^{\sigma\rho} + \Lambda^{\beta}{}_{\sigma}\Lambda^{\alpha}{}_{\rho}T^{\sigma\rho} \\ &= \Lambda^{\alpha}{}_{\rho}\Lambda^{\beta}{}_{\sigma}\left(T^{\rho\sigma} + T^{\sigma\rho}\right) \\ &= 2\Lambda^{\alpha}{}_{\rho}\Lambda^{\beta}{}_{\sigma}T^{(\rho\sigma)} . \end{aligned} \tag{A.51}$$

- Partial differentiation takes a $\binom{p}{q}$ tensor to a $\binom{p}{q+1}$ tensor. For example, under a Lorentz transformation

$$\frac{\partial T^{\alpha}{}_{\beta}}{\partial x^{\mu}} \to \frac{\partial T'^{\alpha}{}_{\beta}}{\partial x'^{\mu}} = \Lambda_{\mu}{}^{\rho}\Lambda^{\alpha}{}_{\sigma}\Lambda_{\beta}{}^{\tau}\frac{\partial T^{\sigma}{}_{\tau}}{\partial x^{\rho}} , \tag{A.52}$$

  and $\partial_{\mu}T^{\alpha}{}_{\beta}$ is a $\binom{1}{2}$ tensor.

Finally, we return to the Minkowski metric. We now see that this is a $\binom{0}{2}$ tensor that takes the same form in all Lorentz frames. This follows since $(\mathbf{\Lambda}^{-1})^T\boldsymbol{\eta}\mathbf{\Lambda}^{-1} = \boldsymbol{\eta}$ implies

$$\eta_{\mu\nu} = \Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}\eta_{\rho\sigma} . \tag{A.53}$$

Similarly, $\eta^{\mu\nu}$ is a $\binom{2}{0}$ tensor and

$$\eta^{\mu}{}_{\nu} = \eta^{\mu\rho}\eta_{\rho\nu} = \delta^{\mu}_{\nu} \tag{A.54}$$

is of type $\binom{1}{1}$.