Mathematical Tripos Part II — Lent term 2020

General Relativity, Examples sheet 4 — Dr U Sperhake

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1 A static space-time has line element

$$ds^2 = -e^{2\phi/c^2} c^2 dt^2 + h_{ij} dx^i dx^j \qquad (i, j = 1, 2, 3)$$

where ϕ and h_{ij} are independent of t. Show that

$$\Gamma^0{}_{\alpha\beta} = \frac{1}{c^2}\left(V_\alpha \frac{\partial\phi}{\partial x^\beta} + V_\beta \frac{\partial\phi}{\partial x^\alpha}\right) \text{ and } \Gamma^i{}_{00} = h^{ij}\frac{\partial\phi}{\partial x^j} e^{2\phi/c^2}$$

where $V_\alpha = (1, 0, 0, 0)$.

Let u^α be the 4-velocity of a co-moving observer (i.e. an observer at rest in these coordinates, so that $u^i = 0$ and $u^0 u_0 = -c^2$). Show that

$$\nabla_\beta u_\alpha = -\frac{1}{c^2} u_\beta \nabla_\alpha \phi$$

and deduce that $\nabla_\alpha \phi = u^\beta \nabla_\beta u_\alpha$. Show further that

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = R_{\alpha\beta}u^\alpha u^\beta$$

and hence that

$$h^{ij}\nabla_i\nabla_j\phi + \frac{1}{c^2}h^{ij}\nabla_i\phi\nabla_j\phi = R_{\alpha\beta}u^\alpha u^\beta$$

[The Ricci identity is $u_{\alpha;\beta\gamma} - u_{\alpha;\gamma\beta} = R^\delta{}_{\alpha\beta\gamma}u_\delta$.] What does this reduce to in the Newtonian limit with $T_{\alpha\beta} = \rho u_\alpha u_\beta$?

2 A perfect fluid has 4-velocity u^α and particle number density n, density ρ and pressure p. The particle flux density N^α and energy-momentum tensor $T^{\alpha\beta}$ are given by

$$N^\alpha = n u^\alpha, \qquad T^{\alpha\beta} = (\rho + p/c^2)u^\alpha u^\beta + p g^{\alpha\beta},$$

and both are conserved: $N^\alpha{}_{;\alpha} = T^{\alpha\beta}{}_{;\beta} = 0$.

(i) Suppose first that the fluid has zero pressure. Show that the fluid flow lines (integral curves of u^α) are geodesics and that ρ is proportional to n on each such geodesic.

(ii) Now consider a general perfect fluid and the weak-field metric

$$ds^2 = -e^{2\varphi/c^2}\, c^2 dt^2 + dx^2 + dy^2 + dz^2\ ,$$

with $\varphi/c^2 \sim v^2/c^2 \ll 1$, where v is a typical speed, so that $u^\alpha \approx (1, \mathbf{u})$. Show that, to lowest order,

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{u}) = 0,$$

where $\boldsymbol{\nabla}$ is the usual 3-dimensional flat space derivative. What is the corresponding equation for ρ? [Recall (sheet 2) that $\Gamma^\beta{}_{\beta\alpha} = \frac{1}{2}(\log(-g))_{,\alpha}$.]

Show that

$$(\rho + p/c^2)u_{\alpha;\beta}u^\beta + p_{,\alpha} + c^{-2}p_{,\beta}u^\beta u_\alpha = 0$$

and hence that, in the Newtonian limit, $\rho u_{i;\beta}u^\beta = -p_{,i}$ $(i = 1, 2, 3)$ i.e. $\dfrac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}\phi - \dfrac{1}{\rho}\boldsymbol{\nabla}p$.

3 The Friedman-Lemaître-Robertson-Walker (FLRW) metric is given by

$$ds^2 = -c^2 dt^2 + a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)$$

and

$$G_{tt} = \frac{3(\dot{a}^2 + kc^2)}{a^2}, \quad c^2 G_{rr} = -\frac{2a\ddot{a} + \dot{a}^2 + kc^2}{1 - kr^2}.$$

For a dust universe ($T_{tt} = \rho c^4$), show that $\rho a^3 = \rho_0$, where ρ_0 is a constant.

(i) In the case $k = 0$, show that $a\dot{a}^2 = A^2$, where A is a constant and deduce that the universe expands for ever. Without further calculation, explain how this conclusion is affected in the case $k < 0$.

(ii) In the case $k > 0$, we define a new coordinate η by $\dfrac{d\eta}{dt} = \dfrac{c}{Ra}$, where $R^2 = k^{-1}$. Derive the equations

$$a(\eta) = B(1 - \cos \eta) \qquad ct(\eta) = BR(\eta - \sin \eta),$$

where B is a constant. Hence show that the universe recollapses within a finite time. Now set $r = R \sin \chi$ in the line element and use the formula for the 3-space volume element

$$dV = \sqrt{g_{\chi\chi} g_{\theta\theta} g_{\phi\phi}}\, d\chi\, d\theta\, d\phi$$

to determine the volume of the universe at a given scale factor (the angular coordinates run from 0 to π for χ and θ, and from 0 to 2π for ϕ). Hence find the maximum volume in terms of MG, where M is the total mass of the universe, and c.

4 Obtain the geodesic equations for the closed ($k = 1$) FLRW dust universe, using η, χ, θ, ϕ coordinates and show that there are null geodesics with $\theta = \chi = \frac{1}{2}\pi$. How many times can a photon encircle the universe from the time of creation to the moment of annihilation?

5 Show that the Einstein-Maxwell equations (i.e. the Einstein equations with energy momentum tensor for an electromagnetic field $T^{\alpha\beta} = F^{\alpha\gamma} F^{\beta}{}_{\gamma} - \frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} g^{\alpha\beta}$) can be written

$$R_{\alpha\beta} = \kappa \big(F_{\alpha\gamma} F_{\beta}{}^{\gamma} - \tfrac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \big).$$

You are given that, for a line element of the form

$$ds^2 = -f(r) c^2 dt^2 + \frac{1}{f(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta\, d\phi^2),$$

the only non-zero components of the Ricci tensor are

$$c^{-2} R_{tt}/f = -f R_{rr} = \tfrac{1}{2} f'' + f'/r, \quad R_{\theta\theta} = R_{\phi\phi}/\sin^2 \theta = 1 - rf' - f.$$

In the case

$$F_{tr} = -F_{rt} = \frac{Q}{r^2}, \text{ with } F_{\alpha\beta} = 0 \text{ otherwise},$$

show that a solution can be found that reduces to the Schwarzchild solution for $Q = 0$.

Find an analogous solution in the case $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$.

6 A spacecraft is freely falling radially into a Schwarzschild black hole. It has 4-velocity V^α and proper time τ. It emits monochromatic radio of wavelength λ_e. Its signals propagate radially outwards and are received, with wavelength λ_o, by a distant observer who is at rest with respect to the Schwarzschild coordinates.

A retarded time coordinate u is defined by $u = ct - r^*$ where $dr/dr^* = F(r)$ and $F(r) = 1 - 2M/r$. Show that

$$ds^2 = -F\,du^2 - 2du\,dr + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,.$$

Show that

$$\frac{\lambda_o}{\lambda_e} = \frac{\Delta t_o}{\Delta\tau} = \frac{\Delta u_o}{c\Delta\tau} = \frac{\Delta u_e}{c\Delta\tau} \approx V^u/c$$

where, for example, Δt_o is the proper time interval during which the observer receives one cycle of the signal and $\Delta\tau$ is the time for the spacecraft to emit one cycle.

Show next that $V_u = -K$, where K is a constant, and that

$$V^u = \frac{K + \sqrt{K^2 - Fc^2}}{F}, \qquad V^r = -\sqrt{K^2 - Fc^2}.$$

Deduce that on the world line of the spacecraft near the horizon $du/dr \sim -2/F$, and that $u \sim -2r^*$ and $F \sim e^{-u/(4M)}$.

Conclude that, just as the transmitter is about to cross the event horizon, the observer sees the frequency red-shifted with an observer-time dependence $\propto \exp(-ct/(4M))$.

7 Show that, for an observer with proper time τ moving in the Schwarzschild space-time,

$$c^2 = Fc^2\dot{t}^2 - \dot{r}^2/F - r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2),$$

where $\dot{t} = dt/d\tau$ etc., and $F = 1 - 2M/r$. Show, that for an observer within the Schwarschild horizon, $\dot{r}^2 \geq -c^2F$ however the observer moves. Deduce that any observer crossing the Schwarzschild horizon will reach $r = 0$ within a proper time $\pi M/c$.

8 Let M be the torus $(S^1 \times S^1)$ and define the metric $g_{\alpha\beta}$ on M by

$$ds^2 = \sin\theta(d\phi^2 - d\theta^2) + 2\cos\theta d\theta d\phi,$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$. Show that, for a null geodesic,

$$\dot{\phi}^2 + 2\dot{\phi}\dot{\theta}\cot\theta - \dot{\theta}^2 = 0,$$

where dot is differentiation with respect to an affine parameter, and deduce that the curves given by $\phi = -2\ln\sin(\theta/2) + \phi_0$ and $\phi = -2\ln\cos(\theta/2) + \phi_0$ are null geodesics. Use another first integral of Lagrange's equations to show that in both cases $\theta = p\lambda$, where λ an affine parameter and p is a constant.

Show that one family of null geodesics wraps round the torus an infinite number of times within a finite affine parameter, never reaching the null curve $\theta = 2\pi$, and that the other family of null geodesics crosses this curve.

Is this space geodesically complete? Is the Riemann tensor well-behaved (no calculation required)?

9 A weak gravitational field has the spacetime metric $g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta} + O(\epsilon^2)$, where $\eta_{\alpha\beta}$ is the Minkowski metric and ϵ small. Show that

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\epsilon[h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}] + O(\epsilon^2).$$

Let $h = h^{\gamma}{}_{\gamma}$ and define $\overline{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}h\,\eta_{\alpha\beta}$. Show that $h_{\alpha\beta} = \overline{h}_{\alpha\beta} - \frac{1}{2}\overline{h}\,\eta_{\alpha\beta}$. Show also that

$$R_{\alpha\beta} = \tfrac{1}{2}\epsilon[\,-\Box\overline{h}_{\alpha\beta} + \overline{h}_{\alpha}{}^{\gamma}{}_{,\beta\gamma} + \overline{h}_{\beta}{}^{\gamma}{}_{,\alpha\gamma} + \tfrac{1}{2}\eta_{\alpha\beta}\Box\overline{h}\,] + O(\epsilon^2).$$

where $\Box = \eta^{\alpha\beta}\nabla_\alpha\nabla_\beta$. What are the linear vacuum equations for $\overline{h}_{\alpha\beta}$?

An infinitesimal coordinate transformation (which may be called a gauge transformation) is given by $x^\alpha \to x^\alpha + \epsilon f^\alpha(x)$. Show that

$$h_{\alpha\beta} \to h_{\alpha\beta} - f_{\alpha,\beta} - f_{\beta,\alpha} + O(\epsilon),$$

but that the curvature tensors are unchanged (to leading order in ϵ). Deduce that if f^α is chosen to satisfy $\Box f^\alpha = \overline{h}^{\alpha\beta}{}_{,\beta}$, then in the new coordinates $\overline{h}^{\alpha\beta}{}_{,\beta} = 0$. Conclude that the linearised Einstein equation for weak fields in vacuum is the wave equation

$$\Box\overline{h}_{\alpha\beta} = 0\,.$$

Consider a *gravitational wave* $h_{\alpha\beta} = H_{\alpha\beta}\,e^{ik_\beta x^\beta}$ in the above gauge, where $H_{ab,c} = 0$. (Note: we really mean $h_{\alpha\beta} \propto H_{\alpha\beta}$ here unlike in the lecture where we started setting $\overline{h}_{\alpha\beta} \propto H_{\alpha\beta}$). Show that $H_{\alpha\beta}k^\beta = \frac{1}{2}k_\alpha H_\beta{}^\beta$ and that k^α is null. Show also that through remaining gauge freedom there is arbitrariness in $H_{\alpha\beta} \to H_{\alpha\beta} + k_\alpha v_\beta + v_\alpha k_\beta$ for any v_α. How many degrees of freedom are there for a gravitational wave propagating in a given direction?

Show that $R_{\alpha\beta\gamma\delta}k^\delta = 0$ to lowest order in ϵ.

If $k^\alpha = k(1,0,0,1)$, show that we may take the independent components to be $H_{11} = -H_{22},\ H_{12} = H_{21}$.