

4.4 Flow past a sphere at low Reynolds number

Uniform flow \mathbf{U} past a fixed rigid sphere, radius a . There are several methods, all of which have heavy algebra somewhere.

4.4.3 Method 3

The linearity of the Stokes equations means that $\mathbf{u}(\mathbf{x})$ must be linear in \mathbf{U} . Further, the problem has spherical symmetry about the centre of the sphere, which is taken as the origin. The velocity and pressure fields must therefore take the forms

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= \mathbf{U}f(r) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})g(r), \\ p(\mathbf{x}) &= \mu(\mathbf{U}\cdot\mathbf{x})h(r),\end{aligned}$$

where $r = |\mathbf{x}|$, and f , g and h are functions of scalar r to be determined.

Now

$$\frac{\partial u_i}{\partial x_j} = U_i x_j f'/r + \delta_{ij} U_n x_n g + x_i U_j g + x_i x_j U_n x_n g'/r.$$

Contracting i with j , we have the incompressibility condition

$$0 = \nabla \cdot \mathbf{u} = U_n x_n (f'/r + 4g + r g').$$

Differentiating again

$$\begin{aligned}\mu \nabla^2 u_i &= \mu U_i (f'' + 2f'/r + 2g) + \mu x_i U_n x_n (g'' + 6g'/r) \\ \nabla_i p(\mathbf{x}) &= \mu U_i h + \mu x_i U_n x_n h'/r\end{aligned}$$

Hence the governing equations give

$$f'/r + 4g + r g' = 0, \quad f'' + 2f'/r + 2g = h \quad \text{and} \quad g'' + 6g'/r = h'/r.$$

Eliminating h and then f yields

$$r^2 g''' + 11r g'' + 24g' = 0.$$

This differential equation is homogeneous in r so there are solutions of the form $g = r^\alpha$. Substituting, one finds $\alpha = 0, -3$ and -5 , with associated $f = -(\alpha + 4)r^{\alpha+2}/(\alpha + 2) + f_0$ and $h = -(\alpha + 5)(\alpha + 2)r^\alpha$. Hence the general solution of the assumed form linear in \mathbf{U} is

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= \mathbf{U} \left(-2Ar^2 + B + Cr^{-1} - \frac{1}{3}Dr^{-3} \right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x}) \left(A + Cr^{-3} + Dr^{-5} \right), \\ p(\mathbf{x}) &= \mu(\mathbf{U}\cdot\mathbf{x}) \left(-10A + 2Cr^{-3} \right).\end{aligned}$$

We shall need the stress exerted across a spherical surface with unit normal $\mathbf{n} = \mathbf{x}/r$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mu \mathbf{U} \left(-3Ar + 2Dr^{-4} \right) + \mu \mathbf{x} (\mathbf{U} \cdot \mathbf{x}) \left(9Ar^{-1} - 6Cr^{-4} - 6Dr^{-6} \right)$$

Applying the boundary conditions on the rigid sphere and for the far field, we find the coefficients

$$A = 0, \quad B = 1, \quad C = -\frac{3}{4}a \quad \text{and} \quad D = \frac{3}{4}a^3,$$

so

$$\mathbf{u} = \mathbf{U} \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3} \right) + \mathbf{x} (\mathbf{U} \cdot \mathbf{x}) \left(-\frac{3a}{4r^3} + \frac{3a^3}{4r^5} \right),$$

$$p = -\frac{3a\mu \mathbf{U} \cdot \mathbf{x}}{2r^3}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n}|_{r=a} = \frac{3\mu}{2a} \mathbf{U}.$$

Hence the drag on the sphere is

$$\int_{r=a} \boldsymbol{\sigma} \cdot \mathbf{n} dS = 4\pi a^2 \frac{3\mu}{2a} \mathbf{U} = 6\pi\mu a \mathbf{U}.$$