Mathematical Tripos Part II **Michaelmas 2026**
D23 Fluid Dynamics II **Prof. J.R. Lister**
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Example Sheet 2

Every answer should include at least one relevant sketch

1. Show that in Stokes flow in unbounded viscous fluid, two identical dense spheres, with centres aligned at an arbitrary angle to vertical, fall under gravity at constant separation, i.e. neither separating nor coming closer together.

2. An external force is applied at the centre of a cube in a direction normal to one of its faces. Using symmetry, show that in unbounded Stokes flow the cube moves in the direction of the applied force. Show also that it does not rotate. Using linearity, deduce that in any orientation the cube (assumed of uniform density) sediments vertically and without rotating.

Using similar arguments [*Which bits are different?*], show that when the cube simply rotates (without translation) about an axis through its centre, the resisting hydrodynamic torque is parallel to the angular velocity and the hydrodynamic force is zero.

*State which of the above results apply to a regular tetrahedron? What about an ellipsoid?

3. If the strain-rate tensor $\mathbf{e}(\mathbf{x})$ vanishes throughout a connected region, show that the flow there is rigid-body motion. [*Hint: Take the curl of* $\mathbf{e}(\mathbf{x})$ *to show first that the vorticity is uniform (constant). Then consider a particular integral plus an irrotational flow.*]

Show that if the surface stress is specified on a bounding surface then the Stokes flow in the interior is unique to within the addition of a rigid-body motion. What condition(s) must the prescribed surface stress satisfy for there to exist a possible Stokes flow in the interior? [*Hint: In the absence of body forces the Stokes equation can be written* $\partial\sigma_{ij}/\partial x_j = 0$.]

4. If $\mathbf{A}(\mathbf{x})$ is a vector harmonic function, i.e. $\nabla^2\mathbf{A} = \mathbf{0}$, show that the flow

$$\mathbf{u} = \boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{x}) - 2\mathbf{A}, \quad p = 2\mu\boldsymbol{\nabla}\cdot\mathbf{A}$$

is incompressible and satisfies the Stokes equation with no body force. Calculate the stress tensor.

For a sphere of radius a translating at velocity $\mathbf{V}$ through a fluid that is otherwise at rest, explain why the harmonic function takes the form

$$\mathbf{A} = \alpha a\mathbf{V}\frac{1}{r} + \beta a^3(\mathbf{V}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\frac{1}{r}.$$

[*Hint: How many vector harmonic functions that are linear in* $\mathbf{V}$ *can you construct using the fundamental harmonic solution* $1/r$ *and its derivatives?*]

Find the values of the constants α and β.

5. Consider an unbounded Stokes flow outside a rigid sphere of radius a rotating with angular velocity $\boldsymbol{\Omega}$. Show that the pressure gradient is zero. Then derive the velocity field as

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\Omega}\wedge\mathbf{x}\frac{a^3}{r^3}\cdot$$

[*Hint: How many* true *vector harmonic functions,* $\mathbf{u}$*, can you construct using the fundamental harmonic solution* $1/r$ *and its derivatives that are linear in the* pseudo*vector* $\boldsymbol{\Omega}$*?*]

Show that the torque exerted on the sphere by the flow is $-8\pi\mu a^3\boldsymbol{\Omega}$.

6. Consider a stationary spherical bubble of radius a in a uniform flow $\mathbf{U}$. Recall from lectures the argument that a Stokes flow linear in $\mathbf{U}$ outside a sphere is of the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{U}f(r) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})g(r).$$

Applying boundary conditions on $r = a$ of no normal component of velocity and no tangential stress, find the flow $\mathbf{u}(\mathbf{x})$. Show that the drag force is $4\pi\mu a\mathbf{U}$. [*You may use any results from the handout on Stokes flow past a rigid sphere.*]

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7. Using the minimum dissipation theorem by stating carefully the flows you are comparing, and exploiting the result from question 5, find upper and lower bounds for the hydrodynamic torque on a regular tetrahedron rotating about its centre in a viscous fluid.

[*Hint: The radius of the inscribed sphere for a regular tetrahedron of edge length a is $a/\sqrt{24}$ while that of the circumscribed sphere is $\sqrt{6}a/4$.*]

8. An incompressible, viscous fluid is contained in the two-dimensional region $-\alpha < \theta < \alpha$ between two rigid hinged plates rotating with equal and opposite angular velocity of magnitude ω. Therefore, in plane polar coordinates, the velocity components on the hinged plates satisfy

$$u_r = 0, \qquad u_\theta = \mp\omega r \quad \text{on} \quad \theta = \pm\alpha.$$

Neglecting all inertial forces, show that a solution to the Stokes problem is of the form

$$\psi = \tfrac{1}{2}\omega r^2 g(\theta)$$

(why?) and find the function $g(\theta)$. Deduce the pressure field $p(r,\theta)$. Is the logarithmic divergence of the pressure an issue for the couple required? What is the physical interpretation of the singularity in g?

9. An incompressible, viscous fluid occupies the region $0 < \theta < \alpha$, $0 < r < \infty$ in plane-polar coordinates (r,θ). It is bounded by a stationary, rigid plate at $\theta = \alpha$ and a rigid plate at $\theta = 0$ that translates with constant velocity U in its own plane in the negative r direction. Calculate the resulting Stokes flow of the fluid. Calculate the stresses on each of the plates and comment on the external forces required to sustain the flow.

10. A spherical annulus of incompressible viscous liquid of volume V occupies the region $R_1(t) < r < R_2(t)$ between two free surfaces on which pressures $P_1(t)$ and $P_2(t)$, respectively, are applied by inviscid interior and exterior fluids. The resulting flow is spherically symmetric. Show (neglecting inertia, gravity and surface tension) that

$$\frac{d}{dt}\left(R_1^3\right) \;=\; \frac{\pi(P_1 - P_2)}{\mu V} R_1^3 \left(R_1^3 + \frac{3V}{4\pi}\right).$$

[*Hints: $u_r = f(t)/r^2$ (why?) and $\sigma_{rr} = -p + 2\mu\,\partial u_r/\partial r$ in this flow. Be careful to distinguish between normal stress and pressure in the viscous fluid.*]

Show that if $P_1 - P_2$ is maintained constant and positive, then R_1 becomes infinite in a finite time. [*Hint: Simplify the notation before solving.*] What happens if $P_1 - P_2$ is maintained constant and negative?

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