Mathematical Tripos Part II **Michaelmas 2025**
D23 Fluid Dynamics II **Prof. J.R. Lister**
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Example Sheet 3

Every answer should include at least one relevant sketch

1. A rigid sphere of radius a falls under gravity through a fluid of viscosity μ towards a horizontal rigid plane. Use lubrication theory to show that, when the minimum gap h_0 is very small, the speed of approach of the sphere is

$$h_0 W/6\pi\mu a^2,$$

where W is the weight of the sphere corrected for buoyancy. How does h_0 vary with time?

2. A fluid of viscosity μ is forced by a pressure difference Δp through the narrow gap between two parallel circular cylinders of radius a with axes $2a + b$ apart. Show that, provided $b \ll a$ and $\rho b^3 \Delta p \ll \mu^2 a$, the volume flux through the gap (per unit length along the cylinders) is approximately

$$\frac{2b^{5/2}\Delta p}{9\pi a^{1/2}\mu}$$

when the cylinders are fixed.

Show that when the two cylinders rotate with angular velocities Ω_1 and $-\Omega_2$ about their respective axes the change in the volume flux [*hint: linearity*] is given by

$$\frac{2}{3}ab(\Omega_1 + \Omega_2).$$

3. A viscous fluid coats the outer surface of a cylinder of radius a which rotates with angular velocity Ω about its axis, which is horizontal. The angle θ is measured from the horizontal on the rising side. Use lubrication theory to show that the volume flux per unit length $Q(\theta, t)$ is related to the thickness $h(\theta, t) \ll a$ of the fluid layer by

$$Q = \Omega a h - \frac{g}{3\nu} h^3 \cos\theta,$$

and deduce an evolution equation for $h(\theta, t)$.

Consider now the possibility of a steady state with $Q = \text{const}$, $h = h(\theta)$. Show that a steady solution with $h(\theta)$ continuous and 2π-periodic exists only if

$$\Omega a > (9Q^2 g/4\nu)^{1/3}.$$

[*Hint: Consider a graph of* $\cos\theta$ *as a function of* h.]

4. A two-dimensional viscous drop of height $h(x, t)$ and extent $-x_N(t) < x < x_N(t)$ spreads under gravity on a horizontal surface. Assuming that the drop has become a thin layer, obtain the partial differential equation governing the evolution of h. Use the boundary conditions at $\pm x_N$ [and the absence of a source at $x = 0$] to show that the (two-dimensional) volume A of the gravity current is constant.

Deduce the form of a possible similarity solution, and determine the corresponding shape and extent of the current. [Your answer for $x_N(t)$ may include the constant $I = \int_{-1}^{1}(1 - s^2)^{1/3} ds$.]

[Use the similarity scalings to estimate when the current can be treated as a viscous thin-film flow.]

5. A fluid with viscosity μ flows in a shallow container with a free surface at $z = 0$. The base of the container is rigid and is located at $z = -h(x, y)$. An external horizontal stress $\mathbf{S}(x, y)$ is applied at the free surface to drive the flow (and gravity keeps the free surface flat at leading order). Using lubrication theory, show that the two-dimensional horizontal volume flux $\mathbf{q}(x, y) \equiv \int_{-h}^{0} \mathbf{u}_H \, dz$, where $\mathbf{u}_H = (u_x, u_y)$, satisfies the equations

$$\boldsymbol{\nabla} \cdot \mathbf{q} = 0, \quad \mu\mathbf{q} = -\frac{1}{3}h^3 \boldsymbol{\nabla} p + \frac{1}{2}h^2 \mathbf{S},$$

where $p(x, y)$ is the modified pressure [i.e. the deviation from $p_0 - \rho g z$]. Find also an expression for the surface velocity $\mathbf{u}_0(x, y) \equiv \mathbf{u}_H(x, y, 0)$ in terms of $\mathbf{S}$, $\mathbf{q}$ and h.

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6. The walls of a straight two-dimensional channel are porous and separated by a distance d. Fluid of viscosity μ is driven along the channel by a pressure gradient $G = -\partial p/\partial x$. At the same time, suction is applied to one wall of the channel providing a cross flow with uniform transverse component of velocity V, with fluid being supplied at this rate through the other wall. Calculate the steady velocity and vorticity distributions in the fluid. Sketch them (i) when $Vd/\nu \ll 1$ and (ii) when $Vd/\nu \gg 1$.

7. Fluid of viscosity μ fills the annulus $a < r < b$ between a long cylinder $r = a$ rotating at angular velocity Ω and a long stationary cylinder $r = b$. Looking up the components of the Navier–Stokes equation in cylindrical coordinates, find the axisymmetric velocity field, ignoring end effects.

Suppose now that the two cylinders are porous, and a pressure difference is applied so that there is a radial flow $-Va/r$ in the fluid annulus. Find an expression for the new steady flow around the cylinder when $Va/\nu \neq 2$. Comment on the flow structure when $Va/\nu \gg 1$.

Find the torque (per unit length along the cylinder) required to maintain the motion, and show that it is independent of both b and μ in the limit $Va/\nu \to \infty$. [Check the dimensions and sign of your result.]

8. Starting from the Navier–Stokes equations for incompressible viscous flow with conservative forces, obtain the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}.$$

Interpret the terms in the equation.

At time $t = 0$ a line vortex is created along the z-axis, with the same circulation Γ around the axis at each z. The fluid is viscous and incompressible, and for $t > 0$ has only an azimuthal velocity denoted v. Show that there is a similarity solution of the form $rv(r)/\Gamma = f(\eta)$, where $r = (x^2 + y^2)^{1/2}$ and η is a suitable similarity variable. Furthermore, show that all conditions are satisfied by

$$f(\eta) = \frac{1}{2\pi}(1 - e^{-\eta^2}), \qquad \eta = \frac{r}{2\sqrt{\nu t}}.$$

Show also that the total vorticity in the plane $z = 0$ remains constant at Γ for all $t > 0$. Sketch v as a function of r.

9. Calculate the vorticity $\boldsymbol{\omega}$ associated with the velocity field

$$\mathbf{u} = (-\alpha x - yf(r,t),\ -\alpha y + xf(r,t),\ 2\alpha z)\,,$$

where α is a positive constant, and $f(r,t)$ depends on $r = (x^2+y^2)^{1/2}$ and time t. Show that the velocity field represents a dynamically possible motion if $f(r,t)$ satisfies

$$2f + r\frac{\partial f}{\partial r} = A\gamma(t)e^{-\gamma(t)r^2},$$

where

$$\gamma(t) = \frac{\alpha}{2\nu}\left(1 \pm e^{-2\alpha(t-t_0)}\right)^{-1},$$

and A and t_0 are constants.

Show that, in the case where the minus sign is taken, γ is approximately $1/[4\nu(t - t_0)]$ when t only just exceeds t_0. Which terms in the vorticity equation dominate when this approximation holds?

10.** [Not to be done at the expense of previous questions] Consider the toy problem $\epsilon y'' + y' + e^{-x} = 0$, $y(0) = 0$ and $y \to 0$ as $x \to \infty$ $(*)$ for $\epsilon \ll 1$.

(i) By solving at successive powers of ϵ, find the first three terms of an asymptotic expansion $y(x) \sim \sum_{i=0}^{\infty} \epsilon^i y_i(x)$ that satisfies $y' + e^{-x} = -\epsilon y''$ with $y \to 0$ as $x \to \infty$.

(ii) By solving at successive powers of ϵ, find the first three terms of an asymptotic expansion $Y(\xi) \sim \sum_{i=0}^{\infty} \epsilon^i Y_i(\xi)$ that satisfies $Y'' + Y' + \epsilon e^{-\epsilon\xi} = 0$ with $Y(0) = 0$. [Note each Y_i has one free constant since there is only one boundary condition.]

(iii) Now let $x = \epsilon\xi$ and let $y(x) = Y(\xi)$ be the solution to the problem $(*)$. Determine the constants in the Y_i by requiring $Y(\xi)$ and $y(x)$ to have the same functional form when $\xi \gg 1$ and $x \ll 1$, respectively.

(iv) Solve $(*)$ directly and verify that your answers in parts (i) and (ii) are the asymptotic expansions of the direct analytic solution as $\epsilon \to 0$ with x fixed and as $\epsilon \to 0$ with ξ fixed respectively.

[This method of 'matched asymptotic expansions' underpins Prandtl's boundary-layer theory, but has many other applications.]

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