**Mathematical Tripos Part II** **Michaelmas 2025**
**D23 Fluid Dynamics II** **Prof. J.R. Lister**
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**Example Sheet 4**

**Every answer should include at least one relevant sketch**

**1. Vorticity.** From the vorticity equation, derive the equation satisfied by the streamfunction $\psi(r,\theta)$ for a steady two-dimensional flow with kinematic viscosity $\nu$ in polar coordinates. Show further that this equation has solutions of the form $\psi = Qf(\theta)$, with $Q$ constant, if $f$ satisfies an ordinary differential equation which you should determine.

Consider flow between rigid walls at $\theta = \pm\alpha$ driven by a point sink of strength $2\alpha Q$ at the origin. State appropriate boundary conditions for the above differential equation for $f$. If $R \equiv Q/\nu \gg 1$ the solution has the form of boundary layers on the walls and an inviscid potential flow in the interior. By making suitable approximations to the differential equation find the potential flow, the scaling for the width of the boundary layer and the leading-order equation within the boundary layer. [You do not need to solve this equation.]

**2. Vorticity.** Write down the equation satisfied by the vorticity $\omega(x,y,t)$ in a two-dimensional flow in Cartesian coordinates. Introduce a streamfunction $\psi$ and show that $\omega = -\nabla^2\psi$. Show that the vorticity equation has a time-dependent similarity solution of the form

$$\psi = CxH(t)^{-1}\phi(\eta), \quad \omega = -CxH(t)^{-3}\phi_{\eta\eta}(\eta), \quad \text{for } \eta = yH(t)^{-1},$$

if $H(t) = (2Ct)^{1/2}$ and if $\phi$ satisfies an ordinary differential equation which you should determine involving an effective Reynolds number, $R \equiv C/\nu$.

**3. Boundary layers.** Wind blowing over a deep reservoir exerts at the water surface a uniform tangential stress, $S$, which is normal to, and away from, a straight side of the reservoir. Use dimensional analysis, based on (a) balancing the inertial and viscous forces in a thin boundary layer and (b) on the imposed boundary condition, to find order-of-magnitude estimates $\delta(x)$ for the boundary-layer thickness and $U(x)$ for the surface velocity as functions of distance $x$ from the shore. Using the boundary-layer equations, find the ordinary differential equation governing the non-dimensional function $f$ defined by

$$\psi(x,y) = U(x)\delta(x)f(\eta), \quad \text{where} \quad \eta = y/\delta(x).$$

What are the boundary conditions on $f$?

**4. Boundary layers.** A steady two-dimensional jet of fluid runs along a plane rigid wall. Use the boundary-layer equations to show that the quantity

$$P = \int_0^\infty u(y)\left(\int_y^\infty u(y')^2\,dy'\right)dy$$

is independent of the distance $x$ along the wall. Find order-of-magnitude estimates for the boundary-layer thickness and velocity as functions of $x$.

Using the notation of question 3, one can derive a third-order differential equation for $f(\eta)$. State conditions sufficient to obtain the appropriate solution to this equation. [Note: the no-slip condition $f'(0) = 0$ is satisfied automatically as a consequence of other conditions.]

*Show that for the analogous axisymmetric wall jet spreading out radially the velocity varies like $r^{-3/2}$.

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**5. Instabilities.** A vortex sheet of strength $U$ is located at a distance $h$ above a rigid wall $y = 0$ and is parallel to it, so that the fluid velocity $(u, 0, 0)$ is

$$u = \begin{cases} U & \text{in } 0 < y < h, \\ 0 & \text{in } y > h\ . \end{cases}$$

Suppose now that the sheet is perturbed slightly to the position $y = h + \mathcal{R}\left[\eta_0 e^{ik(x-ct)}\right]$, where $k > 0$ is real but $c$ may be complex. Show that

$$c = U/(1 \pm i\sqrt{\tanh kh}).$$

Deduce that: (i) the sheet is unstable to disturbances of all wavelengths; (ii) for short waves ($kh \gg 1$) the growth rate $k\,\mathrm{Im}(c)$ is $\frac{1}{2}Uk$ and the wave propagation speed $\mathrm{Re}(c)$ is $\frac{1}{2}U$; (iii) for long waves ($kh \ll 1$) the growth rate is $Uk\sqrt{kh}$ and the propagation speed is $U$. Comment on the effect of the wall on the instability.

**6. Instabilities.** A two-dimensional jet in the $x$-direction has velocity profile

$$u = \begin{cases} 0 & \text{in } y > h, \\ U & \text{in } -h < y < h, \\ 0 & \text{in } y < -h\ . \end{cases}$$

The vortex sheets at $y = \pm h$ are perturbed to

$$y = \begin{cases} +h + \mathcal{R}\left[\eta_1 e^{ik(x-ct)}\right], \\ -h + \mathcal{R}\left[\eta_2 e^{ik(x-ct)}\right]. \end{cases}$$

Show that the jet is unstable to a 'varicose' instability for which $\eta_1 = -\eta_2$ (identical to that of question 5), and also to a 'sinuous' instability for which $\eta_1 = \eta_2$ and

$$c = U/(1 \pm i\sqrt{\coth kh}).$$

**7. Instabilities.** Two regions of the same inviscid fluid are separated by a thin membrane at $y = 0$. The fluid in $y > 0$ has the uniform velocity $(U, 0, 0)$ in cartesian coordinates, while the fluid at $y < 0$ is at rest. The membrane is now slightly perturbed to $y = \eta(x, t)$. The dynamical effect of the membrane is to induce a pressure difference across it equal to $-\beta\partial^4\eta/\partial x^4$, where $\beta > 0$ is a constant (so the pressure is higher below the interface when $\partial^4\eta/\partial x^4 > 0$).

Assuming that the flow remains irrotational and all perturbations are small, derive the relation between $\sigma$ and $k$ for a disturbance of the form $\eta(x, t) = \mathcal{R}\left[Ce^{ikx+\sigma t}\right]$, where $k > 0$ is real but $\sigma$ may be complex. Show that there is an instability only for $k < k_{\max}$ where $k_{\max}$ is to be determined. Find the maximum growth rate and the value of $k$ for which this is obtained.

**8. Free surface flows.** Show that the rate of dissipation of mechanical energy in an incompressible fluid is $2\mu e_{ij}e_{ij}$ per unit volume, where $e_{ij}$ is the rate-of-strain tensor and $\mu$ is the dynamic viscosity.

A finite mass of incompressible fluid, of dynamic viscosity $\mu$ and density $\rho$ is held in the shape of a sphere $r < a$ by surface tension. It is set into a mode of small oscillations in which the velocity field may be taken to have Cartesian components

$$u = \beta x, \quad v = -\beta y, \quad w = 0,$$

where $\beta \propto \exp(-\epsilon t)\sin\omega t$. Assuming that $\epsilon \ll \omega$, calculate the dissipation rate averaged over a cycle (ignoring the slowly varying factor $\exp(-\epsilon t)$) and hence show that $\epsilon = 5\mu/\rho a^2$. You may assume that the total energy of the oscillation is twice the kinetic energy averaged over a cycle. Why is it permissible to ignore the details of the boundary layer near $r = a$?

**9. Free surface flows.** Consider an incompressible potential flow $\mathbf{u} = \boldsymbol{\nabla}\phi$ in a volume $V$ with surface $S$ and outward normal $\mathbf{n}$. Show that

$$2\mu \int_V e_{ij}e_{ij}\, dV = \mu \int_S \boldsymbol{\nabla}\left(|\boldsymbol{\nabla}\phi|^2\right) \cdot \mathbf{n}\, dS.$$

Hence show that the drag on a rising spherical bubble is $-12\pi\mu a\mathbf{U}$.



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