

Example Sheet 4: Nonlinear Waves

Parts of the old Tripos questions 6 and 9 overlap with earlier questions on this sheet. Where this is the case, you may simply quote the earlier results rather than rederive them.

1. *Shock formation.* At time $t = 0$ the velocity $u(x, t)$ in a one-dimensional simple wave, propagating in the positive x direction through a perfect gas, has the form $u = u_m \sin kx$, where u_m and k are positive constants. Find the time t^* at which shocks form. Sketch $u(x)$ at times $t = 0$, $t = \frac{1}{2}t^*$ and $t = t^*$. Show that in the time interval $(0, t^*)$ a single wave-crest (i.e. a local maximum of $u(x, t)$) travels a distance

$$\frac{1}{k} \left(\frac{2c_0}{(\gamma + 1)u_m} + 1 \right).$$

Comment: When $k = 2\pi \times (1\text{kHz})/c_0$, $c_0 = 340\text{ms}^{-1}$, $\gamma = 1.4$, and $u_m = 0.05\text{ms}^{-1}$ (equivalent to 120dB, the pain threshold for the ear), the distance is about 320m.

2. *Shock formation.* A perfect gas, initially at rest, occupies the region to the right of a piston whose position is $X(t) = \frac{1}{2}at^2$ for $t > 0$. Find the time and position where a shock first forms.

3. *Blood flow.* An artery is modelled as a long straight tube with elastic walls and cross-sectional area $A(x, t)$, which contains incompressible, inviscid blood of density ρ . On the assumption that the fluid velocity u and pressure p do not vary across the artery, conservation of mass and momentum imply that

$$A_t + (uA)_x = 0 \quad \text{and} \quad \rho u_t + \rho u u_x = -p_x.$$

The area A is related to the fluid pressure p by an elastic ‘tube law’ of the form $p = P(A)$, where $P(A)$ is some given, strictly increasing function. Find the Riemann invariants and their corresponding propagation speeds.

Now suppose that

$$P(A) = p_0 + \frac{\rho c_0^2}{2\kappa} \left(\frac{A}{A_0} \right)^{2\kappa},$$

where p_0 , A_0 , c_0 and κ are positive constants. For $t < 0$ the artery has uniform area A_0 and there is no flow. Blood is then pumped into the artery ($x > 0$) with velocity $U(t)$ at $x = 0$, where

$$U(t) = \begin{cases} U_0 \frac{t}{t_1} \left(2 - \frac{t}{t_1} \right) & (0 \leq t \leq 2t_1), \\ 0 & (t > 2t_1), \end{cases}$$

and $U_0(1 - \kappa) < c_0$. Calculate the time and place at which a ‘shock’ first forms.

Comment: In an adult human, typical values are $A_0 = 5 \times 10^{-4} \text{m}^2$, $U_0 = 1.2 \text{ms}^{-1}$, $\kappa = 1$, $c_0 = 5 \text{ms}^{-1}$, $p_0 = 10^4 \text{N m}^{-2}$, $\rho = 10^3 \text{kg m}^{-3}$, $t_1 = 0.35 \text{s}$. Do you expect shocks to form?

4. *A general expansion fan.* A piston confines an inviscid compressible fluid (not necessarily a perfect gas) to the right-hand half, $x > 0$, of an infinite tube. The fluid is initially at rest, $u = 0$, with uniform density ρ_0 and sound speed c_0 . For $t > 0$ the piston moves with constant speed V away from the fluid. Assuming that the fluid can keep up with the piston, show that there is a region R_2 in the (x, t) -plane, in which the local sound speed c takes a constant value c_2 , which differs from the value c_0 in the undisturbed region R_0 . Find an equation that determines c_2 in terms of V and the function $c(\rho)$. Deduce the condition on V for the fluid to keep up with the piston.

Show by *reductio ad absurdum*, or otherwise, that all the C_+ characteristics lying outside both R_2 and R_0 must pass through the origin. Deduce that for $t > 0$

$$u + c = \begin{cases} c_2 - V, & -Vt \leq x \leq (c_2 - V)t \\ xt^{-1}, & (c_2 - V)t \leq x \leq c_0 t \\ c_0, & x \geq c_0 t \end{cases}$$

Sketch the forms of u and c as functions of x at two different times.

5. *Expansion fan and escape velocity.* Consider the situation in question 4 for the case of a perfect gas with specific-heat ratio γ . Find the equations in regions R_0 , R_1 and R_2 of

- (i) the C_- characteristic that originates at $x = \xi$ and $t = 0$
- (ii) the trajectory of the gas particle which is at $x = \xi$ when $t = 0$.

Sketch the C_+ and C_- characteristics and the particle trajectories in the (x, t) -plane. Hence explain what happens when $V > 2(\gamma - 1)^{-1}c_0$.

6. *Two expansion fans* (Tripos 75425). A perfect gas, with constant specific heats in the ratio γ , is initially at rest with uniform sound speed c_0 . It is confined by two pistons to the region $0 < x < 2\ell$ of a long cylindrical tube. At time $t = 0$, both pistons are set into impulsive motion away from the gas with constant velocities $u = -V < 0$ and $u = U > 0$.

(i) For $0 \leq t \leq \ell/c_0$ show that in the part $x \leq \ell$ of the tube (which cannot have been reached by any signal from the piston initially at $x = 2\ell$), every C_+ characteristic is a straight line. Show that the fluid velocity u takes the value

$$u = \frac{2}{\gamma + 1} \left(\frac{x}{t} - c_0 \right) \quad \text{for} \quad \left(c_0 - \frac{\gamma + 1}{2}V \right)t < x < c_0t .$$

Give the corresponding value of c . Find the shape of the C_- characteristics when u and c take these values.

(ii) Deduce that, when $t > \ell/c_0$, the equation

$$u = \frac{2}{\gamma + 1} \left(\frac{x}{t} - c_0 \right)$$

is satisfied only in the smaller interval

$$\left(c_0 - \frac{\gamma + 1}{2}V \right)t < x < \frac{\ell}{\gamma - 1} \left((\gamma + 1) \left(\frac{c_0t}{\ell} \right)^{(3-\gamma)/(\gamma+1)} - 2 \left(\frac{c_0t}{\ell} \right) \right) .$$

(iii) For a case with V/c_0 about $\frac{1}{2}$ and U/c_0 about $\frac{1}{4}$, give a rough sketch indicating **four** areas of the (x, t) plane throughout each of which u takes a different constant value, to be specified.

7. *A piston-generated shock.* A piston moves with constant positive velocity u_1 into a perfect gas of specific heat ratio $\gamma > 1$, generating a shock wave which moves ahead of the piston. Show that a possible solution of all the relevant equations is one in which the gas is at rest beyond the shock, at pressure p_0 , and is moving with constant velocity u_1 in the region between the piston and the shock, throughout which region the density and pressure also take constant values ρ_1, p_1 which are determined by

$$\frac{\rho_1}{\rho_0} = \frac{2\gamma + (\gamma + 1)\beta}{2\gamma + (\gamma - 1)\beta}, \quad \frac{1}{\beta^2} + \frac{\gamma + 1}{2\gamma\beta} = \frac{c_0^2}{\gamma^2 u_1^2},$$

where β is the shock strength defined as $(p_1 - p_0)/p_0 > 0$, and ρ_0 and c_0 are the density and sound speed of the undisturbed gas. Show also that the shock speed $V = c_0(1 + \frac{\gamma+1}{2\gamma}\beta)^{1/2}$.

8. *Traffic flow.* Assume that the speed of cars down a long straight (one-way) road is a known, monotonically decreasing function $u(\rho)$ of the local density ρ of traffic. The flux of cars is thus given by $q(\rho) = \rho u$. From conservation of cars deduce that ρ is constant on characteristics $dx/dt = c(\rho)$, where $c = dq/d\rho$. Deduce also that if a shock develops between regions of density ρ_1 and ρ_2 then it propagates with speed $[q(\rho_1) - q(\rho_2)]/(\rho_1 - \rho_2)$.

Consider the case $u(\rho) = U(1 - \rho/\rho_0)$ where U is (10% faster than) the speed limit and ρ_0 is the density of a nose-to-tail traffic jam. Sketch the functions $q(\rho)$ and $c(\rho)$. Explain why shocks only form when light traffic is behind heavy traffic, and why the shocks can travel either forwards or backwards depending on the density of traffic.

A queue of cars with density ρ_0 is waiting in $-L < x < 0$ behind a red traffic light at $x = 0$. There are no other cars on the road. The light turns green at $t = 0$. Find the time T when the last car starts to move, and determine the velocity of the last car for $t > T$. [*Hint:* The solution involves both a shock and an expansion fan.]

9.* A method to generate shock waves in a 'shock tube' (Tripos 85427). An infinitely long uniform tube contains two perfect gases separated by a membrane at $x = 0$. The gas in $x > 0$ has pressure p_1 , density ρ_1 and specific heat ratio γ_1 ; the corresponding values for the gas in $x < 0$ are p_2, ρ_2, γ_2 where $p_2 > p_1$. At $t = 0$ the membrane is burst. Assuming that the interface between the two gases remains plane and moves with constant speed V , use the one-dimensional equations of motion to show that there are three regions in the tube in which the pressure is uniform,

$$\begin{aligned}
 p &= p_2 \quad \text{for } x < - \left(\frac{\gamma_2 p_2}{\rho_2} \right)^{1/2} t, \\
 p &= p_1 \quad \text{for } x > Ut, \\
 p &= p_m \quad \text{for } - \left[\left(\frac{\gamma_2 p_2}{\rho_2} \right)^{1/2} - \frac{\gamma_2 + 1}{2} V \right] t < x < Ut,
 \end{aligned}$$

where p_m is as yet unknown, and the shock velocity, U , is a constant to be found in terms of $p_m, p_1, \rho_1, \gamma_1$.

Show that V is related to p_m by the following two equations:

$$\begin{aligned}
 V &= (p_m - p_1) \left(\frac{1}{2} \rho_1 [(\gamma_1 + 1)p_m + (\gamma_1 - 1)p_1] \right)^{-1/2}, \\
 V &= \frac{2}{\gamma_2 - 1} \left(\frac{\gamma_2 p_2}{\rho_2} \right)^{1/2} \left[1 - \left(\frac{p_m}{p_2} \right)^{(\gamma_2 - 1)/2\gamma_2} \right],
 \end{aligned}$$

and hence show that there is a unique solution for p_m and V .