

## Mathematical Tripos Part II: Michaelmas Term 2023

## Numerical Analysis – Examples' Sheet 2

11. Let  $a(x) > 0$ ,  $x \in [0, 1]$ , be a given smooth function. We solve the diffusion equation with variable diffusion coefficient,  $u_t = (au_x)_x$ , given with an initial condition for  $t = 0$  and boundary conditions at  $x = 0$  and  $x = 1$ ,  $t \geq 0$ , with the finite-difference method

$$u_m^{n+1} = u_m^n + \mu [a_{m-1/2} u_{m-1}^n - (a_{m-1/2} + a_{m+1/2}) u_m^n + a_{m+1/2} u_{m+1}^n],$$

where  $a_s = a(sh)$ ,  $\mu = \frac{\Delta t}{(\Delta x)^2}$ ,  $n \geq 0$ ,  $1 \leq m \leq M$  and  $h = \Delta x = \frac{1}{M+1}$ . Prove that the local error is  $\mathcal{O}(h^4)$ . Then, justifying carefully every step of your analysis, show (e.g. by using the eigenvalue technique) that the method is stable for all  $0 < \mu < \frac{1}{2a_{\max}}$ , where  $a_{\max} = \max_{x \in [0,1]} a(x)$ .

[Hint: In the second half, use Gershgorin theorem to show that the matrix  $A$  occurring in the relation  $u^{n+1} = Au^n$  satisfies  $\rho(A) \leq 1$ .]

12. Apply the Fourier stability test to the difference equation

$$u_m^{n+1} = \frac{1}{2}(2 - 5\mu + 6\mu^2)u_m^n + \frac{2}{3}\mu(2 - 3\mu)(u_{m-1}^n + u_{m+1}^n) - \frac{1}{12}\mu(1 - 6\mu)(u_{m-2}^n + u_{m+2}^n),$$

where  $m \in \mathbb{Z}$ . Deduce that the test is satisfied if and only if  $0 \leq \mu \leq \frac{2}{3}$ .

13. A square grid is drawn on the region  $\{(x, t) : 0 \leq x \leq 1, t \geq 0\}$  in  $\mathbb{R}^2$ , the grid points being  $(m\Delta x, n\Delta x)$ ,  $0 \leq m \leq M+1$ ,  $n = 0, 1, 2, \dots$ , where  $\Delta x = \frac{1}{M+1}$  and  $M$  is odd. Let  $u(x, t)$  be an exact solution of the wave equation  $u_{tt} = u_{xx}$  and let the boundary values  $u(x, 0)$ ,  $0 \leq x \leq 1$ ,  $u(0, t)$ ,  $t > 0$ , and  $u(1, t)$ ,  $t > 0$ , be given. Further, an approximation to  $\partial u / \partial t$  at  $t = 0$  allows each of the function values  $u(m\Delta x, \Delta x)$ ,  $m = 1, 2, \dots, M$ , to be estimated to accuracy  $\epsilon$ . Then, the difference equation

$$u_m^{n+1} = u_{m+1}^n + u_{m-1}^n - u_m^{n-1}$$

is applied to estimate  $u$  at the remaining grid points. Prove that all of the moduli of the errors  $|u_m^n - u(m\Delta x, n\Delta x)|$  are bounded above by  $\frac{1}{2}\epsilon M$ , even when  $n$  is very large.

[Hint: Verify that the local error is zero. For  $n = 1$  and  $1 \leq m \leq M$ , let the error in  $u(m\Delta x, \Delta x)$  be  $\delta_{mk}\epsilon$ , where  $\delta_{mk}$  is the Kronecker delta and where  $k$  is an arbitrary integer in  $(1, 2, \dots, M)$ . Draw a diagram that shows the contribution from this error to  $u_m^n$  for every  $m$  and  $n > 1$ .]

14. A rectangular grid is drawn on  $\mathbb{R}^2$ , with grid spacing  $\Delta x$  in the  $x$ -direction and  $\Delta t$  in the  $t$ -direction. Let the difference equation

$$\begin{aligned} & u_m^{n+1} - 2u_m^n + u_m^{n-1} \\ &= \mu [a(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) + b(u_{m-1}^n - 2u_m^n + u_{m+1}^n) + c(u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1})], \end{aligned}$$

where  $\mu = \frac{(\Delta t)^2}{(\Delta x)^2}$ , be used to approximate solutions of the wave equation  $u_{tt} = u_{xx}$ . Deduce that, with constant  $\mu$ , the local error is  $\mathcal{O}((\Delta x)^4)$  if and only if the parameters  $a$ ,  $b$  and  $c$  satisfy  $a = c$  and  $a + b + c = 1$ . Show also that, if these conditions hold, then the Fourier stability condition is achieved for all values of  $\mu$  if and only if the parameters also satisfy  $|b| \leq 2a$ .

[Hint: In the second half, the roots of the characteristic equation satisfy  $x_1 x_2 = 1$ . Then,  $|x_1|, |x_2| \leq 1$  if  $D \leq 0$ , where  $D$  is the discriminant of the equation.]

15. For a given analytic function  $f$  we consider its truncated Fourier approximation on the interval  $[-1, 1]$ , i.e.,

$$f(x) \approx \phi_N(x) = \sum_{n=-N/2+1}^{N/2} \hat{f}_n e^{i\pi n x}, \quad \text{where } \hat{f}_n = \frac{1}{2} \int_{-1}^1 f(\tau) e^{-i\pi n \tau} d\tau, \quad n \in \mathbb{Z}.$$

Prove that, given any  $s = 1, 2, \dots$ , we have for all  $n \in \mathbb{Z} \setminus \{0\}$  the equality

$$\hat{f}_n = \frac{(-1)^{n-1}}{2} \sum_{m=0}^{s-1} \frac{1}{(\pi i n)^{m+1}} \left[ f^{(m)}(1) - f^{(m)}(-1) \right] + \frac{1}{(\pi i n)^s} \widehat{f^{(s)}}_n.$$

16. Unless  $f$  is analytic, the rate of decay of its Fourier harmonics can be very slow, certainly slower than  $\mathcal{O}(N^{-1})$ . To explore this, let  $f(x) = |x|^{-1/2}$ . Prove that  $\hat{f}_n = g(-n) + g(n)$ , where  $g(n) = \int_0^1 e^{i\pi n \tau^2} d\tau$ . Moreover, with the error function  $\operatorname{erf}$  defined as the integral

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2} d\tau, \quad z \in \mathbb{C}.$$

show that its Fourier coefficients are

$$\hat{f}_n = \frac{\operatorname{erf}(\sqrt{i\pi n})}{2\sqrt{i\pi n}} + \frac{\operatorname{erf}(\sqrt{-i\pi n})}{2\sqrt{-i\pi n}},$$

and asymptotically for  $|n| \gg 1$  we have  $\hat{f}_n = \mathcal{O}(n^{-1/2})$ . [Hint: For the last identity use without proof the asymptotic estimate  $\operatorname{erf}(\sqrt{i\pi x}) = 1 + \mathcal{O}(x^{-1})$  for  $x \in \mathbb{R}$ ,  $|x| \gg 1$ .]

17. Consider the solution of the two-point boundary value problem

$$(2 - \cos \pi x)u'' + u = 1, \quad -1 \leq x \leq 1, \quad u(-1) = u(1),$$

using the spectral method. Plugging the Fourier expansion of  $u$  into this differential equation, show that the  $\hat{u}_n$  obey a three-term recurrence relation. Calculate  $\hat{u}_0$  separately and using the fact that  $\hat{u}_{-n} = \hat{u}_n$  (why?), prove further that the computation of  $\hat{u}_n$  for  $-N/2+1 \leq n \leq N/2$  (assuming that  $\hat{u}_n = 0$  outside this range of  $n$ ) reduces to the solution of an  $(N/2) \times (N/2)$  tridiagonal system of algebraic equations.

18. Set

$$a(x) = \sum_{n=-\infty}^{\infty} \hat{a}_n e^{i\pi n x}, \quad (2.1)$$

the Fourier expansion of  $a$ . Explain why  $a$  is periodic with period 2. Further, let  $\tilde{n}$  denote some selected value of  $n$ . Evaluate  $\frac{1}{2} \int_{-1}^1 a(x) e^{-i\pi \tilde{n} x} dx$  with  $a(x)$  given by (2.1). Doing so, you have just computed the Fourier coefficient  $\hat{a}_{\tilde{n}}$ . Now choose  $a(x) = \cos \pi x$  and compute its corresponding Fourier coefficients. With this, derive an explicit expression for the coefficients in the  $N$ -term truncated Fourier approximation of the solution  $u$  of

$$\begin{cases} ((\cos \pi x + 2)u_x)_x = \sin \pi x, & x \in [-1, 1] \\ \text{periodic boundary conditions and normalisation condition} & \int_{-1}^1 u(x) dx = 0. \end{cases}$$

19. Let  $u$  be an analytic function in  $[-1, 1]$  that can be extended analytically into the complex plane and possesses a Chebyshev expansion  $u = \sum_{n=0}^{\infty} \check{u}_n T_n$ . Express  $u'$  in an explicit form as a Chebyshev expansion.

20. The two-point ODE  $u'' + u = 1$ ,  $u(-1) = u(1) = 0$ , is solved by a Chebyshev method.

- Show that the odd coefficients are zero and that  $u(x) = \sum_{n=0}^{\infty} \check{u}_{2n} T_{2n}(x)$ . Express the boundary conditions as a linear condition of the coefficients  $\check{u}_{2n}$ .
- Express the differential equation as an infinite set of linear algebraic equations in the coefficients  $\check{u}_{2n}$ .
- Discuss how to truncate the linear system, keeping in mind the exponential convergence of the method and the floating-point precision of your computer.
- While  $u(-1) = u(1)$  we cannot expect a standard spectral method to converge at spectral speed. Why?