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Mathematical Tripos Part II: Michaelmas Term 2023

Numerical Analysis – Examples' Sheet 3

21. As discussed in the Lectures periodicity is necessary for spectral convergence. Suppose that an analytic function f on $[-1,1]$ is not periodic, yet $f(-1) = f(+1)$ and $f'(-1) = f'(1)$. Integrating by parts the Fourier coefficients $\hat{f}_n$ show that $\hat{f}_n = \mathcal{O}(n^{-3})$. Show that the rate of convergence of the N-terms truncated Fourier expansion of f is hence $\mathcal{O}(N^{-2})$.

    Now, suppose $f(-1) \neq f(1)$. We can force the values at the endpoints to be equal. Set $f(x) = \frac{1}{2}(1-x)f(-1) + \frac{1}{2}(1+x)f(+1) + g(x)$, where $g(x) = f(x) - \frac{1}{2}(1-x)f(-1) - \frac{1}{2}(1+x)f(+1)$. Verify that $g(\pm 1) = 0$ and that if f is analytic then so is g. The idea is now to represent f as a linear function plus the Fourier expansion of g, i.e.,

    $$f(x) = \frac{1}{2}(1-x)f(-1) + \frac{1}{2}(1+x)f(+1) + \sum_{n=-\infty}^{\infty} \hat{g}_n\, e^{i\pi n x}.$$

    We can iterate this idea: To do so, construct a function h for which $h(\pm 1) = h'(\pm 1) = 0$ and verify that $\hat{h}_n = \mathcal{O}(n^{-3})$. [*Hint: In the second construction the function f will be represented as a cubic function plus the Fourier expansion of h.*]

22. Consider the following boundary value problem for the heat equation

    $$\begin{cases} u_t = u_{xx}, & -1 \le x \le 1,\ t > 0 \\ u(-1,t) = u(1,t),\ u_x(-1,t) = u_x(1,t), & t > 0 \\ u(x,0) = e^{i\pi M x}, & -1 \le x \le 1. \end{cases},$$

    where $M \in \mathbb{Z}$. By separation of variables one can compute the exact solution and get

    $$u(x,t) = e^{-\pi^2 M^2 t}\, e^{i\pi M x}.$$

    Now, approximate the solution u by its N-term truncated Fourier series and solve the spectral approximation for the heat equation, i.e.,

    $$\sum_{n=-N/2+1}^{N/2} \frac{d\hat{u}_n}{dt}(t) e^{i\pi n x} = \sum_{n=-N/2+1}^{N/2} \hat{u}_n(t) \frac{d^2}{dx^2} e^{i\pi n x}.$$

    What do you receive? What is the error of this method with the correct choice of N?

23. By Theorem 4.12, the Gauss-Seidel method for the solution of $A\boldsymbol{x} = \boldsymbol{b}$ converges whenever the matrix A is symmetric and positive definite. Show, however, by a 3×3 counterexample, that the Jacobi method for such an A need not converge. [*Warning: For Jacobi, it is not enough to construct a positive definite A such that $2D - A$ is not positive definite, because we did not prove that the Householder-John theorem gives a criterion. So, you need also to prove that $\rho(D^{-1}(A-D)) > 1$.*]

24. Let the Gauss-Seidel method be applied to the equations $A\boldsymbol{x} = \boldsymbol{b}$ when A is the nonsymmetric 2×2 matrix

    $$A = \begin{bmatrix} 10 & -3 \\ 3 & 1 \end{bmatrix}.$$

    Find the spectral radius of the iteration matrix. Then show that the relaxation method, described in Lecture 17, can reduce the spectral radius by a factor of 2.9. Further, show that iterating twice with Gauss-Seidel with this relaxation decreases the error $\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(\infty)}\|$ by more than a factor of ten. Estimate the number of iterations of the original Gauss-Seidel method that would be required to achieve this decrease in the error.

25. The function $u(x) = x\,(x-1)$, $0 \le x \le 1$, is defined by the equations $u''(x) = 2$, $0 \le x \le 1$, and $u(0) = u(1) = 0$. A difference approximation to the differential equation provides the estimates $u_m \approx u(mh)$, $m = 1, 2, \ldots, M-1$, through the system of equations

$$\begin{cases} u_{m-1} - 2u_m + u_{m+1} = 2h^2, \quad m = 1, 2, \ldots, M-1 \\ u_0 = u_M = 0, \end{cases}$$

where $h = 1/M$, and M is a large positive integer. Show that the exact solution of the system is just $u_m = u(mh)$, $m = 1, 2, \ldots, M-1$.
We employ the notation $u_m^{(\infty)} = u(mh)$, because we let the system be solved by the Jacobi iteration, using the starting values $u_m^{(0)} = 0$, $m = 1, 2, \ldots, M-1$. Prove that the iteration matrix H has the spectral radius $\rho(H) = \cos(\pi/M)$. Further, by regarding the initial error vector $\boldsymbol{u}^{(0)} - \boldsymbol{u}^{(\infty)}$ as a linear combination of the eigenvectors of H, show that the largest component of $\boldsymbol{u}^{(k)} - \boldsymbol{u}^{(\infty)}$ for large k is approximately $(8/\pi^3)\cos^k(\pi/M)$. Hence deduce that the Jacobi method requires about $2.5M^2$ iterations to achieve $\|\boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(\infty)}\|_\infty \le 10^{-6}$.

26. Implement using your favourite language (Matlab, Python, Julia, etc.) the multigrid method as seen in lecture to solve the 1D Poisson equation $u'' = f$ on $[0, 1]$ with zero Dirichlet boundary conditions $u(0) = u(1) = 0$. Try your method on a grid of size $m = 2^{10} - 1$, with a forcing term containing high and low frequencies. Try changing the parameters of the algorithm (Jacobi vs. Gauss-Seidel, etc.) and comment.

27. Apply the standard form of the conjugate gradient method to the linear system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

starting as usual with $\boldsymbol{x}^{(0)} = \mathbf{0}$. Verify that the residuals $\boldsymbol{r}^{(0)}, \boldsymbol{r}^{(1)}$ and $\boldsymbol{r}^{(2)}$ are mutually orthogonal, that the search directions $\mathbf{d}^{(0)}, \mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$ are mutually conjugate, and that $\boldsymbol{x}^{(3)}$ satisfies the equations.

28. Let the standard form of the conjugate gradient method be applied when A is positive definite. Express $\mathbf{d}^{(k)}$ in terms of $\boldsymbol{r}^{(i)}$ and $\beta^{(i)} > 0$, $i = 0, 1, \ldots, k$. Then deduce in a few lines from the formula $\boldsymbol{x}^{(k+1)} = \sum_{i=0}^{k} \omega^{(i)} \mathbf{d}^{(i)}$, from $\omega^{(i)} > 0$, and from the fact that $\boldsymbol{r}^{(i)}$ are orthogonal, that the sequence $\{\|\boldsymbol{x}^{(k)}\| : k = 0, 1, \ldots\}$ increases monotonically.

29. The polynomial $p(x) = x^m + \sum_{i=0}^{m-1} c_i x^i$ is the *minimal polynomial* of the $n \times n$ matrix A if it is the polynomial of lowest degree that satisfies $p(A) = 0$. Note that $m \le n$ holds because of the Cayley-Hamilton theorem.
Give an example of a 3×3 symmetric positive definite matrix with a quadratic minimal polynomial.
Prove that (in exact arithmetic) the conjugate gradient method requires at most m iterations to calculate the exact solution of $A\boldsymbol{v} = \boldsymbol{b}$, where m is the degree of the minimal polynomial of A.