

### Examples Sheet 1

1. (a) Given any fixed  $t_1 < 0 < t_2$ , show that the initial-value problem  $\dot{x} = x^{2/3}$ ,  $x(0) = 0$  has a solution that has  $x(t) = 0$  for  $t \in [t_1, t_2]$ , (and  $x(t) \neq 0$  for any other  $t$ ). Sketch your solution.

The solution to the IVP is clearly not unique. Why can this occur here?

- (b) Solve the differential equation  $\dot{x} = 1 + x^2$  with  $x(0) = x_0$ . Why is the solution unique? Show that for all values of  $x_0$  the solution blows up in finite time both forwards and backwards.

2. By considering only the signs of  $\dot{x}$  and  $\dot{y}$ , i.e. *without* linearizing about the fixed points, sketch the phase portrait for each of the systems in  $\mathbb{R}^2$

$$(i) \quad \dot{x} = x^2 - 1, \quad \dot{y} = y^2 - 1, \quad (ii) \quad \dot{x} = y^2 - 1, \quad \dot{y} = x^2 - 1.$$

[Hint: one of these systems is Hamiltonian.]

3. Use polar coordinates to sketch the phase portrait of

$$\dot{x} = -y(x^2 + y^2), \quad \dot{y} = x(x^2 + y^2).$$

Find the flow  $\phi_t(r_0, \theta_0)$ . Identify the orbit  $\mathcal{O}(\mathbf{x}_0)$  and the  $\omega$ -limit set  $\omega(\mathbf{x}_0)$ .

\*Discuss the stability of the solution  $\mathbf{x}(t)$  with  $\mathbf{x}(0) = \mathbf{x}_0$  and of the orbit  $\mathcal{O}(\mathbf{x}_0)$  with respect to small perturbations of the initial value  $\mathbf{x}_0$ . (Stability might not have been formally defined yet: in that case, instead think about possible definitions.)

4. Consider the system in  $\mathbb{R}^3$

$$\dot{x} = yz, \quad \dot{y} = -zx, \quad \dot{z} = -z^3.$$

Solve the equations by first transforming the  $(x, y)$ -plane to polar coordinates. Then determine  $\phi_t(\mathbf{x}_0)$ . Hence show that the  $\omega$ -limit set for the orbit of  $(x_0, y_0, z_0)$  with  $z_0 \neq 0$  is a circle in the plane  $z = 0$ . Find the flow and  $\omega$ -limit set if  $\dot{z} = -z$  instead of  $-z^3$ .

\*Discuss the stability of all fixed points in both cases.

5. Sketch the phase portrait of the Hamiltonian system

$$\dot{x} = y, \quad \dot{y} = x^2 - 4.$$

The solution with initial condition  $(-4, 0)$  is  $\mathbf{x}(t) = (2 - 6 \operatorname{sech}^2 t, 12 \operatorname{sech}^2 t \tanh t)$ . Show this solution on your phase portrait. What are  $\omega(-4, 0)$ ,  $\omega(-3, 0)$  and  $\omega(-5, 0)$ ?

6. Sketch the phase portraits of the following linear systems, classifying the fixed point at the origin, and finding the eigenvectors only when it helps the sketching:

$$\begin{array}{llll}
 (a) \quad \dot{x} = -2x - 3y, & \dot{y} = 8x + 8y, & (e) \quad \dot{x} = 2x + y, & \dot{y} = 2y, \\
 (b) \quad \dot{x} = x, & \dot{y} = -3x - y, & (f) \quad \dot{x} = 4x + 2y, & \dot{y} = 2x + y, \\
 (c) \quad \dot{x} = 7x - 2y, & \dot{y} = 5x + 5y, & (g) \quad \dot{x} = 5x, & \dot{y} = 5y. \\
 (d) \quad \dot{x} = 7x + 5y, & \dot{y} = -10x - 7y. & & 
 \end{array}$$

For which systems is the origin a hyperbolic fixed point? Which systems are Hamiltonian?

7. Find and classify all fixed points of the system

$$\dot{x} = 2x - x^3 - 3xy^2, \quad \dot{y} = y - y^3 - x^2y.$$

[Hint: the system is invariant under  $x \mapsto -x$  and  $y \mapsto -y$ ; some thought will save yourself some work!] Sketch the phase portrait.

8. Consider the system

$$\dot{x} = -x + \frac{y}{\log \sqrt{x^2 + y^2}}, \quad \dot{y} = -y - \frac{x}{\log \sqrt{x^2 + y^2}}.$$

Show that the origin is a stable focus, in the sense that  $\theta \rightarrow \infty$  as  $t \rightarrow \infty$ , even though the linearized system at the origin is a stable node. [Note that the nonlinear terms are not  $O(|\mathbf{x}|^2)$  here.]

9\*. Sketch the phase portraits near the origin for the following non-hyperbolic fixed points.

$$\begin{array}{llll}
 (a) \quad \dot{x} = x^2, & \dot{y} = y, & (b) \quad \dot{x} = x^2 + xy, & \dot{y} = \frac{1}{2}y^2 + xy, \\
 (c) \quad \dot{x} = y, & \dot{y} = x^2, & (d) \quad \dot{x} = y, & \dot{y} = -x^3 + 4xy.
 \end{array}$$

[Hints: Consider the signs of  $\dot{x}$ ,  $\dot{y}$ . Are the axes trajectories? Is there symmetry? One system is Hamiltonian. Note that case (d) has two exact solutions of the form  $y = a_{\pm}x^2$ .]

10. Calculate the stable and unstable manifolds of the origin to third order (i.e. up to and including the cubic terms) for the system

$$\dot{x} = -x + y^2, \quad \dot{y} = y - x^2.$$

Sketch the phase portrait for  $|\mathbf{x}| \ll 1$  showing the slight curvature of the two manifolds. Find and classify the other fixed point, and sketch the phase portrait on the scale  $|\mathbf{x}| = O(1)$ .

11. Calculate the stable and unstable manifolds of the origin up to fourth order for the system

$$\dot{x} = x + y^2, \quad \dot{y} = -y + 4x^3 + xy.$$

12\*\*. Consider the non-autonomous differential equation  $\dot{y} = y^2 - t$  (which Liouville proved cannot be solved in terms of solutions of algebraic equations or integrals thereof). Convert it to an autonomous system and sketch its portrait in the  $(t, y)$  plane.

Show that solutions that enter the region  $y^2 < t$  cannot leave it. By changing variable from  $y$  to  $u = y/\sqrt{t}$  and considering  $\dot{u}$  for  $t \gg 1$ , explain why all forwards solutions (i.e.  $t$  increasing) tend to  $u = \pm 1$  or  $u = \infty$ .

Argue that all backwards solutions, and all forwards solutions with  $u \rightarrow \infty$ , enter regions where  $\dot{y} > \frac{1}{2}y^2$  and hence blow up in finite time. Explain why there is a unique value  $y(0)$  such that  $y(t)$  remains finite and positive for all finite  $t$  as  $t$  increases and show this solution on your phase portrait.