

Example Sheet 1

1. In the following, the indices i, j, k, ℓ, m take the values 1, 2, 3, and the summation convention applies. In particular let \mathbf{n} be a unit vector, i.e. $n_i n_i = 1$.

- (a) Simplify the following expressions

$$\delta_{ij} a_j, \quad \delta_{ij} \delta_{jk}, \quad \delta_{ij} \delta_{ji}, \quad \delta_{ij} n_i n_j, \quad \epsilon_{ijk} \delta_{jk}, \quad \epsilon_{ijk} \epsilon_{ij\ell}, \quad \epsilon_{ijk} \epsilon_{ikj}, \quad \epsilon_{ijk} (\mathbf{a} \times \mathbf{b})_k.$$

- (b) For each of the following equations, either give the equivalent in vector or matrix notation or explain why the equation is invalid.

$$x_i = a_i b_k c_k + d_i, \quad x_i = a_j b_i + c_k d_i e_k f_j, \quad u = \epsilon_{jkl} v_k w_\ell x_j,$$

$$\epsilon_{ijk} x_j y_k \epsilon_{ilm} x_\ell y_m = \mu, \quad A_{ik} B_{kl} = T_{ik} \delta_{kl}, \quad x_k = A_{ki} B_{ji} y_j.$$

- (c) Write the following equations in suffix notation using the summation convention.

$$(\mathbf{x} + \mu \mathbf{y}) \cdot (\mathbf{x} - \mu \mathbf{y}) = \kappa, \quad \mathbf{x} = |\mathbf{a}|^2 \mathbf{b} - |\mathbf{b}|^2 \mathbf{a}, \quad (2\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{a} + \mathbf{b}) = \lambda.$$

- (d) Given that $A_{ij} = \epsilon_{ijk} a_k$ (for all i, j), show that $2a_k = \epsilon_{kij} A_{ij}$ (for all k).

- (e) Show that $\epsilon_{ijk} s_{ij} = 0$ (for all k) if and only if $s_{ij} = s_{ji}$ (for all i, j).

- (f) Given that $N_{ij} = \delta_{ij} - \epsilon_{ijk} n_k + n_i n_j$ and $M_{ij} = \delta_{ij} + \epsilon_{ijk} n_k$, show that $N_{ij} M_{jk} = 2\delta_{ik}$.

2. A fluid flow has the constant velocity vector (in Cartesian coordinates)

$$\mathbf{v}(\mathbf{r}) = (0, 0, W).$$

Explicitly calculate the volume flux of fluid,

$$Q = \int \mathbf{v} \cdot d\mathbf{S},$$

flowing across (a) the open hemispherical surface $r = a, z \geq 0$, and (b) the disc $r \leq a, z = 0$. Verify that the divergence theorem holds.

3. For a surface S enclosing a volume V , apply the divergence theorem to a vector field $\mathbf{F} = \mathbf{a}p$, where \mathbf{a} is an arbitrary constant vector and $p(\mathbf{r})$ is a scalar field. Deduce that

$$\int_V \nabla p \, dV = \oint_S p \, d\mathbf{S}.$$

4. A time-independent magnetic field $\mathbf{B}(\mathbf{r})$ is given by

$$\mathbf{B} = \frac{\mu_0 I}{2\pi} \frac{\mathbf{e}_z \times \mathbf{r}}{x^2 + y^2},$$

where μ_0 is the magnetic permeability and I is a constant. Using Cartesian coordinates, calculate the electric current density \mathbf{J} given by the steady Maxwell equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Also evaluate $\oint_C \mathbf{B} \cdot d\mathbf{r}$, where C is a circle of radius a in the plane $z = 0$ and centred on $x = y = 0$. Discuss whether Stokes's theorem applies in this situation.

5. Show that, in Cartesian coordinates,

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}).$$

This vector identity remains true for all coordinate systems; however, for non-Cartesian coordinates,

$$(\nabla^2 \mathbf{F})_i \neq \nabla^2 F_i.$$

Why is this the case? Illustrate this point by evaluating $\nabla^2 \mathbf{F}$ for $\mathbf{F} = f(\rho) \mathbf{e}_\phi$ in cylindrical polar coordinates (ρ, ϕ, z) and comparing the result with $\nabla^2 f$.

6. Find the general circularly symmetric solution to the fourth-order equation

$$\nabla^4 \Psi \equiv \nabla^2(\nabla^2 \Psi) = 0.$$

Hint: use plane polar coordinates (ρ, ϕ) , and do not be too eager to expand everything out.

Find those circularly symmetric solutions in the unit disc that are equal to unity at the centre $\rho = 0$ and vanish on the boundary $\rho = 1$. Give a further condition to render the solution unique.

7. Parabolic coordinates (u, v, ϕ) are defined in terms of Cartesian coordinates (x, y, z) by

$$x = uv \cos \phi, \quad y = uv \sin \phi, \quad z = \frac{1}{2}(u^2 - v^2).$$

Show that the surfaces of constant u , and those of constant v , are surfaces obtained by rotating parabolae about the z -axis. What are the surfaces of constant ϕ ? Show that the coordinate surfaces intersect at right angles and hence that these coordinates are orthogonal. Find the scale factors (h_u, h_v, h_ϕ) defined by

$$|d\mathbf{r}|^2 = h_u^2 du^2 + h_v^2 dv^2 + h_\phi^2 d\phi^2.$$

Hence obtain the Laplacian in these coordinates using the formula

$$\nabla^2 = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial q_3} \right) \right].$$

8. Consider the two-stage transformation of Cartesian coordinates (x, y, z) given by

$$\begin{aligned} x &= ax', & y &= by', & z &= cz', \\ x' &= r' \sin \theta' \cos \phi', & y' &= r' \sin \theta' \sin \phi', & z' &= r' \cos \theta', \end{aligned}$$

where a , b and c are positive constants. Calculate the Jacobian matrices of the transformations $(x, y, z) \mapsto (x', y', z')$ and $(x', y', z') \mapsto (r', \theta', \phi')$ and verify explicitly that

$$\frac{\partial(x, y, z)}{\partial(r', \theta', \phi')} = \frac{\partial(x, y, z)}{\partial(x', y', z')} \frac{\partial(x', y', z')}{\partial(r', \theta', \phi')}.$$

Are the coordinates (r', θ', ϕ') orthogonal? What range of these coordinates is required to cover the interior of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1?$$

Express the volume element in coordinates (r', θ', ϕ') and hence calculate the volume of the ellipsoid.

9. In a Cartesian coordinate system (x_1, x_2, x_3) , A is the point $(0, 0, -1)$, B is the point $(0, 0, 1)$ and P is an arbitrary point (x_1, x_2, x_3) . In a curvilinear coordinate system, the coordinates of P are specified by

$$u_1 = \frac{1}{2}(r_1 + r_2) \quad u_2 = \frac{1}{2}(r_1 - r_2) \quad u_3 = \phi,$$

where r_1 and r_2 are the distances AP and BP respectively and ϕ is the angle between the planes ABP and $x_2 = 0$. Show that $x_3 = u_1 u_2$ and that the distance ρ from P to the x_3 -axis is equal to $\sqrt{(u_1^2 - 1)(1 - u_2^2)}$. Next evaluate $\partial x_i / \partial u_j$ (with $i, j = 1, 2, 3$). Deduce that the curvilinear coordinates are orthogonal and sketch the coordinate surfaces. Show that the metric coefficients are

$$h_1 = \sqrt{\frac{u_1^2 - u_2^2}{u_1^2 - 1}}, \quad h_2 = \sqrt{\frac{u_1^2 - u_2^2}{1 - u_2^2}}, \quad h_3 = \sqrt{(u_1^2 - 1)(1 - u_2^2)}.$$

Show that if the function Ψ satisfies Laplace's equation and is independent of u_2 and u_3 then it has the form

$$\Psi = a + b \ln \left(\frac{u_1 - 1}{u_1 + 1} \right)$$

for constant a and b .

10. A uniform stretched string of length L , mass per unit length ρ and tension $T = \rho c^2$ is fixed at both ends. The motion of the string is resisted by the surrounding medium, the resistive force per unit length being $-2\mu\rho\dot{y}$, where $y(x, t)$ is the transverse displacement and $\dot{y} = \partial y / \partial t$. Generalize the argument given in lectures to show that the equation of motion of the string is

$$\frac{\partial^2 y}{\partial t^2} + 2\mu \frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

Find $y(x, t)$ if $\mu = \pi c/L$, $y(x, 0) = d \sin(\pi x/L)$ and $\dot{y}(x, 0) = 0$.

If an extra transverse force $F \sin(\pi x/L) \cos(\pi ct/L)$ per unit length acts on the string, find the resulting forced oscillation.

11. Show that the solution of Laplace's equation, $\nabla^2 \Phi = 0$, in the region $0 < x < a$, $0 < y < b$, $0 < z < c$, with $\Phi = 1$ on the surface $z = 0$ and $\Phi = 0$ on the other surfaces, is

$$\Phi = \frac{16}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin[(2m-1)\pi x/a] \sin[(2n-1)\pi y/b] \sinh[k(c-z)]}{(2m-1)(2n-1) \sinh(kc)},$$

where $k^2 = (2m-1)^2 \pi^2/a^2 + (2n-1)^2 \pi^2/b^2$.

12. The temperature distribution $\theta(x, t)$ along a thin bar of length L satisfies the one-dimensional diffusion equation

$$\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2},$$

where the diffusivity ν is a constant, t denotes time and x is the distance from one of the ends. Find $\theta(x, t)$ if the bar is insulated at each end (i.e. if $\partial\theta/\partial x = 0$ at each end), and if the initial temperature distribution is given by

$$\theta(x, 0) = 2\theta_0 \cos^2\left(\frac{\pi x}{L}\right),$$

where θ_0 is a constant. For large times what is the temperature distribution in the bar? Comment.

This example sheet is available at [http://www.damtp.cam.ac.uk/user/examples/Hints and answers](http://www.damtp.cam.ac.uk/user/examples/Hints%20and%20answers) will be posted to CamTools. Supervisors may request these early. Comments/corrections to M.Wingate@damtp.cam.ac.uk.