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NATURAL SCIENCES TRIPOS **Dr Adrian Kent**

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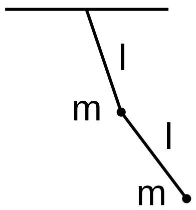
Mathematical Methods III: Small Oscillations and Group Theory
Examples 1

1. Let $T = \frac{1}{2}T_{ij}\dot{q}_i\dot{q}_j$ and $V = \frac{1}{2}V_{ij}q_iq_j$. Verify that the equations of motion $T_{ij}\ddot{q}_j + V_{ij}q_j = 0$ imply that the energy $T + V$ is conserved. Can the constancy of $T + V$ be used to deduce the equations of motion?

2. Consider a model of a linear molecule AAB where the atom B is at one end. The masses are $M_A = m,\ M_B = 2m$ and the spring constants for the forces between neighbouring atoms are given by $k_{AA} = k,\ k_{AB} = 2k$. Find the equations of motion and the normal frequencies for linear oscillations along the axis of the molecule. Verify the orthogonality relation for the normal mode vectors $\mathbf{Q}^{(m)}$ and find the normalized form of these generalized eigenvectors.

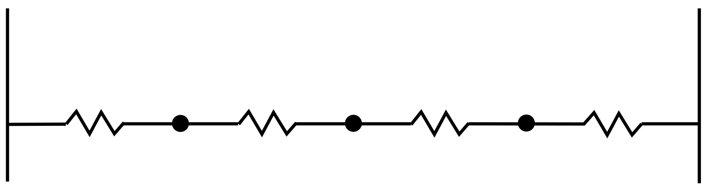
3.



Consider the double pendulum shown, with massless rods, which is constrained to move in a fixed vertical plane. Find the two normal modes of small oscillations and their frequencies.

(It is easier to work out the kinetic and potential energies and then use Lagrange's equations, rather than work out Newton's equations).

4.



Three equal masses are connected by equal springs as shown, the walls being fixed. Find the normal modes of oscillation along the line of the springs, and the ratios of the normal frequencies.

5. Find the group table for the cyclic group C_4, consisting of rotations in a plane by angles $\frac{1}{2}n\pi$, where $0 \leq n \leq 3$. Find the group table for the so-called Vierergruppe V, consisting of the identity and the rotations by π about the x, y and z axes in three-dimensional space. Show that both of these groups are abelian, but that they are not isomorphic.

6. The symmetry group of an N-gon is generated by a single rotation by $2\pi/N$, denoted R, and a reflection m (being any single one of the possible reflectional symmetries). Show by means of sketches that the relations $R^N = I$, $m^2 = I$ and $Rm = mR^{-1}$ are always obeyed. Deduce that the elements of the group are R^n and $R^n m$ $(1 \leqslant n \leqslant N)$. Use the foregoing relations to express the product of two arbitrary elements of the group either in the form R^n or in the form $R^n m$.

7. The six functions $f_1, f_2, \ldots, f_6$ are defined by

$$f_1(z) = z,\ f_2(z) = (1-z)^{-1},\ f_3(z) = 1 - z^{-1},$$
$$f_4(z) = z^{-1},\ f_5(z) = 1 - z,\ f_6(z) = z(z-1)^{-1}.$$

Show that these functions form a group under function composition, i.e., 'function of a function': the 'product' $f_1 f_2$ being defined by $f_1 f_2(z) \equiv f_1(f_2(z))$. Construct the group table, find all subgroups, and determine which of them are normal. (There are three order-2 subgroups none of which is normal, and one order-3 subgroup, which is normal.) Is this group isomorphic to any other group mentioned in the lectures?

What happens to the points $z = 0, 1, \infty$ for each of the functions $f_1, f_2, \ldots, f_6$?

8. $GL(n, \mathbb{R})$ is the group of all invertible $n \times n$ real matrices, and $\mathbb{R}^*$ the multiplicative group of non-zero real numbers. Show that the map $\Phi : GL(n, \mathbb{R}) \to \mathbb{R}^*$ defined by $\Phi(M) = \det M$ is a homomorphism. What is the kernel of Φ? Show that the kernel is a normal subgroup of $GL(n, \mathbb{R})$ and describe its cosets. Show that the product of two cosets is well defined and that it produces another coset. Deduce that the set of all cosets forms a group. (This is an example of a 'quotient group'.)

9. Show that the order of a permutation P is the lowest common multiple of the orders of its component cycles. Resolve

$$P = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 4\,6\,9\,7\,2\,5\,8\,1\,3 \end{pmatrix}$$

into cycles and find its order.

10. A permutation $P \in \sum_N$ may be regarded as permuting the components of an N-vector, and is then represented by a permutation matrix.
For instance the permutation (12) (34) is represented by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Show that the determinant of a permutation matrix is either 1 or -1, and show that the permutations whose permutation matrices have determinant 1 form a subgroup of $\sum_N$ with $\frac{1}{2}N!$ elements. Verify that this is the subgroup of even permutations.

11. Consider the group of translations and rotations in two dimensions acting on the complex variable z by $z \to az + b$ where $|a| = 1$. Show that the matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

form a faithful representation. Show that, within the 2-dimensional space of column vectors with 2 entries, there is an invariant subspace.