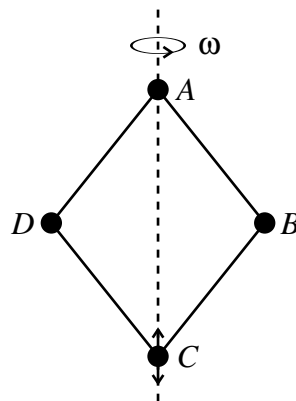


**Example Sheet 1**

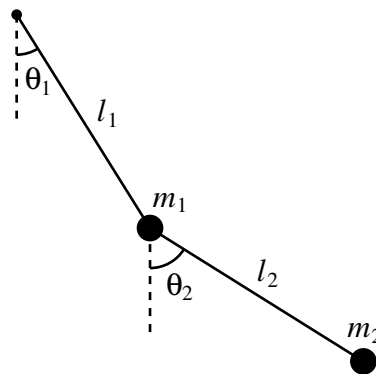
1. Practice in applications of variational calculus [*optional, and certainly not at the expense of other questions*].
  - (a) Show that the shortest path between two points in Euclidean space is a straight line segment.
  - (b) Show that the geodesics (i.e. the shortest paths between two points) on a spherical surface are arcs of great circles. [*Hint: A great circle is the intersection of a sphere with a plane containing the centre of the sphere.*]
  - (c) The problem of the *brachistochrone* (meaning ‘shortest time’) is as follows. Given two points in a vertical plane, find the path in that plane from the higher point to the lower one that minimizes the time taken by a particle to slide under gravity, without friction, starting from rest. Show that the solution is an arc of an inverted cycloid with a cusp at the point from which the particle is released. [*Hint: The differential equation of a cycloid generated by a circle of radius  $a$  is  $(dy/dx)^2 = (2a/y) - 1$ .*]
  
2. Four equal light rods of length  $l$  are hinged together to form a rhombus  $ABCD$ , which lies in a vertical plane. Each of the vertices has mass  $m$ . The vertex  $A$  is fixed, while  $C$  lies directly beneath it and is free to slide up and down. The whole system rotates with constant angular velocity  $\omega$  around the vertical axis  $AC$ .



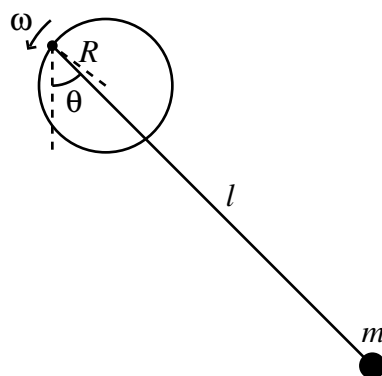
Identify suitable generalized coordinate(s) and write down the Lagrangian of the system.

3. A circular hoop of radius  $a$  lies in a vertical plane. The hoop rotates with constant angular velocity  $\omega$  around a fixed vertical axis that goes through its centre,  $O$ . A bead of mass  $m$  is threaded on the hoop and moves without friction. Its location is denoted by  $A$ . The angle between the line  $OA$  and the downward vertical is  $\psi(t)$ .

- (a) Using the Lagrangian formalism, derive a second-order differential equation for  $\psi(t)$ .
- (b) Derive the same differential equation using the Newtonian formalism. Compare the two methods.
- (c) Assume now that the hoop rotates freely about the vertical axis without friction. Write down the Lagrangian of the system, neglecting the mass of the hoop. Find the additional conserved quantity.
4. A double pendulum is drawn below. Two light rods, of lengths  $l_1$  and  $l_2$ , oscillate in the same plane. Attached to them are masses  $m_1$  and  $m_2$ . How many degrees of freedom does the system have? Write down the Lagrangian describing the dynamics. Derive the equations of motion.



5. The pivot of a simple pendulum is attached to the rim of a disc of radius  $R$ , which rotates about its centre in the plane of the pendulum with constant angular velocity  $\omega$ . (See the diagram below.) Write down the Lagrangian and derive the equation of motion for the dynamical variable  $\theta$ .



6. A particle moves in one dimension in a potential  $V(x)$ , where  $x$  is the spatial coordinate. The dynamics is governed by the Lagrangian

$$L = \frac{1}{12}m^2\dot{x}^4 + m\dot{x}^2V - V^2.$$

Show that the resulting equation of motion is identical to that which arises from the more traditional Lagrangian,  $L = \frac{1}{2}m\dot{x}^2 - V$ .

7. The Lagrangian for a relativistic point particle of mass  $m$  is

$$L = -mc^2 \sqrt{1 - \frac{|\dot{\mathbf{r}}|^2}{c^2}} - V(\mathbf{r}),$$

where  $c$  is the speed of light. Derive the equation of motion, and show that it reduces to Newton's equation of motion in the limit  $|\dot{\mathbf{r}}| \ll c$ .

8. An electron, of mass  $m$  and charge  $-e$ , moves in a magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{r})$ . The Lagrangian for the motion is

$$L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - e\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}).$$

Show that Lagrange's equations reproduce the Lorentz force law for the electron. Then:

- (a) With respect to cylindrical polar coordinates  $(r, \theta, z)$ , consider the vector potential

$$\mathbf{A} = \frac{f(r)}{r} \mathbf{e}_\theta,$$

where  $\mathbf{e}_\theta$  is the unit vector in the  $\theta$  direction. At some initial time, the electron is at a distance  $r_0$  from the  $z$  axis; its velocity is then in the  $(r, z)$  plane. Show that the electron's angular velocity about the  $z$  axis is given by

$$\dot{\theta} = \frac{e}{mr^2} [f(r) - f(r_0)].$$

- (b) (Again, with respect to cylindrical polar coordinates.) Consider the (different) vector potential,

$$\mathbf{A} = rg(z) \mathbf{e}_\theta,$$

where  $g(z) > 0$ . Find two constants of the motion. The electron is projected from a point  $(r_0, \theta_0, z_0)$  with velocity  $(2er_0g(z_0)/m) \mathbf{e}_\theta$ . Show that the electron will then describe a circular orbit, provided that  $g'(z_0) = 0$ . Show that this orbit is stable against small translations in the  $z$  direction, provided that  $g''(z_0) > 0$ .

9. A particle of mass  $m_1$  is restricted to move on a circle of radius  $R_1$  in the plane  $z = 0$ , with centre at  $(x, y) = (0, 0)$ . A second particle, of mass  $m_2$ , is restricted to move on a circle of radius  $R_2$  in the plane  $z = c$ , with centre at  $(x, y) = (0, a)$ . The two particles are connected by a spring; the resulting potential energy is

$$V = \frac{1}{2}\omega^2 d^2,$$

where  $d$  is the distance between the particles.

- (a) Identify the two generalized coordinates and write down the Lagrangian of the system.
- (b) Write down the Lagrangian in the case that one circle lies directly beneath the other,  $a = 0$ , and identify a conserved quantity that appears in this case.
10. Two particles, each of mass  $m$ , are connected by a light rope of length  $l$ . One particle moves on a smooth horizontal table at a variable distance  $r$  from a hole, through which the rope is threaded. The second particle hangs beneath the table.
- (a) Assume initially that the second particle hangs directly beneath the hole. Write down the Lagrangian of the system in terms of  $r$  and a variable  $\psi$ , describing the angle that the first particle makes with respect to a fixed axis. Identify an ignorable coordinate. Write down the equation of motion for the remaining coordinate, assuming that the rope remains taut.
- (b) Assume now that the second particle oscillates beneath the table, as a spherical pendulum. How many degrees of freedom does the system now have? Write down the Lagrangian describing this motion, assuming that the rope remains taut at all times. How many ignorable coordinates are there?
11. Consider a system with  $n$  dynamical degrees of freedom, and generalized coordinates denoted by  $q^a$ , with  $a = 1, \dots, n$ . The most general form for a purely kinetic Lagrangian is

$$L = \frac{1}{2} g_{ab}(q^1, \dots, q^n) \dot{q}^a \dot{q}^b, \quad (*)$$

where the summation convention is being used. The functions  $g_{ab} = g_{ba}$  depend on the generalized coordinates. Assume that  $\det(g_{ab}) \neq 0$  so that the inverse matrix  $g^{ab}$  exists (obeying  $g^{ab} g_{bc} = \delta^a_c$ ). Show that Lagrange's equations for this system are given by

$$\ddot{q}^a + \Gamma_{bc}^a \dot{q}^b \dot{q}^c = 0, \quad (\dagger)$$

where one defines

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} \left( \frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d} \right).$$

[*Remark:* The functions  $g_{ab}$  define a *metric* on the configuration space, and the equations ( $\dagger$ ) are known as the *geodesic equations*. In addition to appearing naturally in differential geometry, these equations arise in general relativity, describing the motion of a particle falling freely under gravity (where a gravitational field is described by a curved spacetime). Lagrangians of the form (\*), known as *sigma models*, appear in many other areas of physics, such as the study of solids, of nuclear forces and of string theory.]

Please send any comments and corrections to [gio10@cam.ac.uk](mailto:gio10@cam.ac.uk)