Note on a Controlled Interconversion Between Two Minimal Surfaces

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Recent work [Raufaste, *et al.*, *Soft Matter* **18**, 4944 (2022)] studied dynamics of a soap film in the shape of an unstable minimal surface whose evolution is governed in part by the frictional forces associated with surface Plateau border (SPB) motion. In this note, we study a variant of this problem in which a half-catenoid bounded by a wire loop and a fluid bath axisymmetrically surrounds a cylindrical rod with a radius equal to the neck of the critical catenoid given by the wire loop. When the half-catenoid is brought just beyond the point of instability, the film touches the cylinder and separates from the bath, creating an SPB that is dragged upwards along the rod by the now unstable soap film, and asymptotically relaxes to a new stable annular minimal surface. We find that the SPB motion is consistent with a theoretical analysis in which the surface tension force associated with the contracting soap film balances the frictional force $f \sim Ca^{2/3}$ given by Bretherton's law, where Ca is the capillary number.

Key words: Minimal surfaces, surface Plateau border, Bretherton's Law

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1. Introduction

Starting from the work of Courant (1940), there has been significant interest in the dynamics of interconversions between soap film minimal surfaces triggered by boundary perturbations. These include Courant's original paradigm of the interconversion between a Möbius strip and a disc, an example for which later work (Goldstein *et al.* 2010, 2014; Pesci *et al.* 2015; Machon *et al.* 2016) discovered that the topological rearrangement is associated with a singularity at the film's boundary that involves reconnection of the associated surface Plateau border (SPB). Because, at least at its early stage, the dynamics of the instability involves a competition between inertial and capillary forces (Keller & Miksis 1983), the film motion is in the regime of high Reynolds number and is very rapid.

As a means of slowing down such topological rearrangements, we recently introduced (Goldstein *et al.* 2021; Raufaste *et al.* 2022) a version of the classical instability of a catenoidal soap film in which the rotational symmetry is broken by introducing a surface cutting the catenoid so that the motion involves SPBs moving along the surface, providing the bulk of the viscous dissipation. In that work, we found that the speed of the moving SPB was quantitatively consistent with a balance between capillary forces and viscous dissipation within the SPB, obeying Bretherton's law (Bretherton 1961) in which the viscous force f per unit length scales as $f = A\gamma Ca^{2/3}$, where A is a dimensionless

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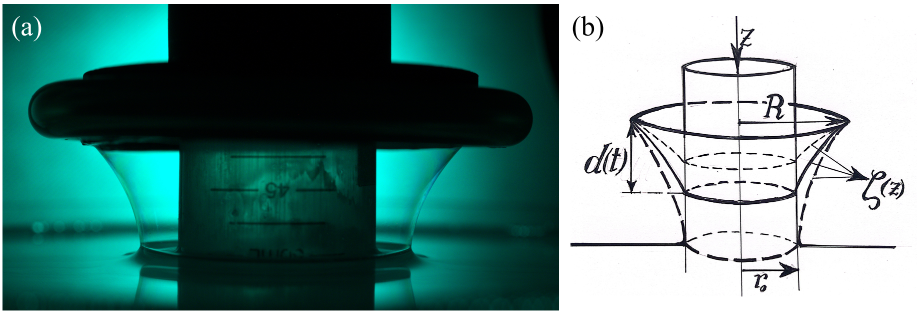

FIGURE 1. Experimental setup. (a) A half-catenoid spanning a loop and a fluid bath and surrounding a cylinder whose radius equal the critical neck radius of the catenoid defined by the loop. (b) Schematic of setup as the soap film retracts.

constant, γ is the surface tension, and the capillary number is $Ca = \mu v/\gamma$, with μ the fluid viscosity and v the SPB velocity.

A simple configuration to study the migration of an SPB was proposed in section IV of Moffatt *et al.* (2016). This involved placing a rod symmetrically along the axis of a catenoidal soap film suspended by two circular wires drawn slowly apart just beyond the separation of critical stability. The collapsing soap film then impacts the rod. and splits into two parts with SPBs propagating in opposite directions along the rod boundary. In the above paper, an estimate for the SPB velocity was obtained by dimensional analysis.

In the present paper, we refine this model in a manner that allows experimental realisation and control. We again do so in the context of a catenoid brought to its point of criticality, by introducing a coaxial central cylinder whose radius is chosen to correspond to the neck of the critical catenoid supported by the loop. We take advantage of the fact (Salkin *et al.* 2014) that a soap film minimal surface connecting a circular support to a soap solution below is exactly a half-catenoid (Fig. 1(a)). Thus, by slowly raising the supporting loop the film evolves through a continuum of stable half-catenoids with progressively smaller neck radii, until it reaches the critical state and touches the cylinder. Beyond the critical state, an SPB disconnects from the bath and moves upward under the action of the film's surface tension and resisted by dissipation within the SPB, eventually rising to the level of the upper loop to form a stable minimal surface in the form of an annulus. This setup therefore achieves a controlled interconversion between two minimal surfaces: the half-catenoid and the annular disk. In the following, we describe a theoretical approach to this dynamical process and its comparison with experiments.

2. Theory and Experimental Verification

The geometry of the setup is shown in Fig. 1(b): A wire loop of radius R held at a distance less than $d_{\max}$ above a soap solution ($d_{\max} = 0.663R$ is the critical height of a half-catenoid) supports a soap film whose shape is the function $\zeta(z)$ surrounding a central cylinder of radius $r_0 = 0.553R$, the corresponding critical neck radius. The initial condition of the surface is the critical half-catenoid where the contact line of the surface with the cylinder coincides with the surface of the bath. When the loop is moved beyond $d_{\max}$ the SPB detaches from the bath and begins its motion up the cylinder.

While the motion of the soap film is a moving boundary problem in which the shape of the film is to be determined as part of the dynamics, we explore the simplest possibility, a quasistatic approximation in which, at every instant of time, the moving film is taken to be an (unstable) equilibrium catenoid that connects the present location of the SPB

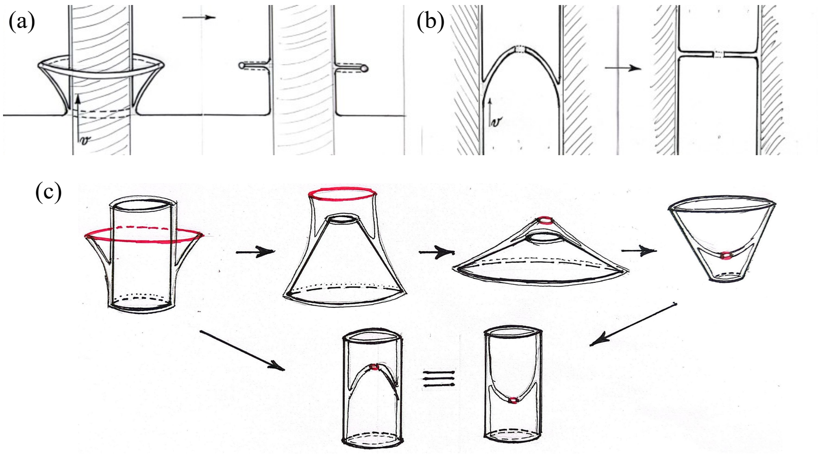

FIGURE 2. Geometry of Plateau borders for two topologically equivalent situations. (a) Relaxation of a half-catenoid. (b) Relaxation of a bubble with an embedded ring in a tube. (c) Pathway of smooth interconversion between (a) and (b).

to the wire loop above. The motion is then governed by a balance between the film's capillary force and the viscous drag within the moving SPB.

We adopt a coordinate system with the origin on the plane of the loop, with positive z downwards, as in Fig. 1(b). Let $d(t)$ be the time-dependent SPB location measured from the origin, $z = 0$. Under the quasistatic approximation the shape of the moving surface can be found by a standard analysis. The general form of the catenoid is

$$\zeta(z) = a \cosh\left(\frac{z-c}{a}\right), \tag{2.1}$$

where the constants a and c are determined by the boundary conditions. These are $\zeta(0) = R$ and, for all $d \equiv d(t) \leqslant d_{\max}$, $\zeta(d) = r_0$. We adopt a system of units made dimensionless with the loop radius R and define

$$\alpha = \frac{a}{R}, \qquad \beta = \frac{r_0}{R}, \qquad D = \frac{d}{R}, \tag{2.2}$$

and find from the boundary conditions the result

$$\beta = \cosh\left(\frac{D}{\alpha}\right) - \sqrt{1-\alpha^2}\,\sinh\left(\frac{D}{\alpha}\right), \tag{2.3}$$

which can be solved to yield a transcendental equation relating D and α,

$$D_{\pm} = \alpha \ln\left[\frac{\beta \pm \sqrt{\beta^2-\alpha^2}}{1-\sqrt{1-\alpha^2}}\right]. \tag{2.4}$$

These two branches combine to give a loop in the αD-plane. The contact angle θ at the SPB, defined to be 0 when the film is tangent to the cylinder, is given by

$$\theta = -\tan^{-1}\left[\sinh\left(\frac{D_-}{\alpha} - \cosh^{-1}\left(\frac{1}{\alpha}\right)\right)\right]. \tag{2.5}$$

The motion of the soap film arises from a balance between the capillary force $\gamma\cos\theta$ per unit length of SPB, with θ given by (2.5), and frictional forces that arise from flows within the Plateau border. The latter have been considered previously by Cantat *et al.* (2004);

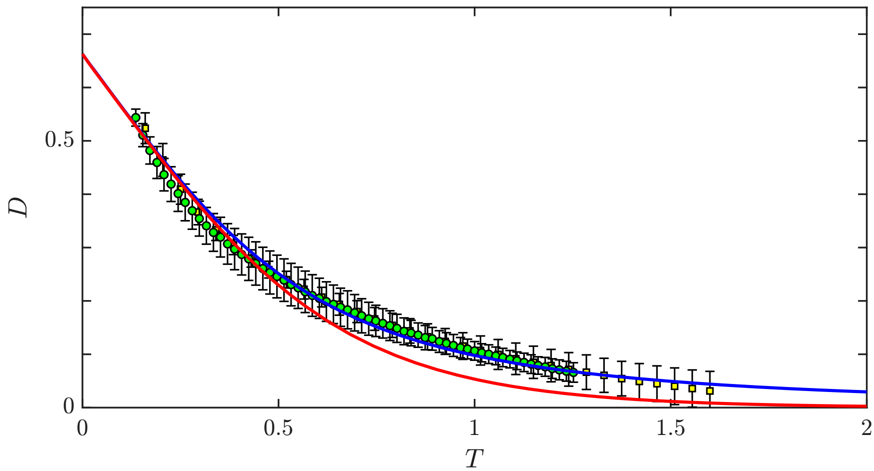

FIGURE 3. Time dependence of the SPB position in scaled coordinates. Data points are from experiments, blue line is the solution to (2.7) for $q = 3/2$, and red line is for $q = 1$.

Cantat & Delannay (2005); Terriac *et al.* (2006), who noted in steady-state problems that Bretherton's law holds. As shown in Fig. 2, the Plateau border problems of a moving bubble in a tube and the present situation are related by topological inversion. Thus we expect that per unit length of the contact line between the film and the central cylinder there is a frictional force of the form previously used to describe the motion of a bubble in a tube, with a general power-law exponent q,

$$f = A\left(\frac{\mu v}{\gamma}\right)^q, \tag{2.6}$$

where A is a dimensionless constant and $q = 2/3$ corresponds to Bretherton's law. For the case of interest ($q = 2/3$), we define a rescaled time $T = \gamma t/(\mu R A^{1/q})$, and then the scaled SPB position obeys the equation

$$\frac{dD}{dT} = -\cos(\theta(D))^{1/q} = -\left(\frac{\alpha(D)}{\beta}\right)^{1/q}, \tag{2.7}$$

where the second relation follows from (2.5) by straightforward but lengthy algebra, and $\alpha(D)$ is given implicitly by (2.4). While it is possible to obtain an implicit solution for $T(\alpha)$ in terms of hypergeometric functions for any power $q \leqslant 1$, the result is not particularly illuminating and a numerical solution is straightforward. The function $D(T)$ is shown in Fig. 3 for $q = 2/3, 1$. One can easily show that the asymptotic behaviour of D for large T is $D \sim T^{-2}$ when $q = 2/3$, which contrasts strongly with exponential decay obtained when the frictional law is taken to be linear in the capillary number ($q = 1$).

To test which power law exponent q governs the SPB motion, we employed an apparatus consisting of a 50 ml Falcon tube (diameter 39 mm) attached by its threaded end to a large plastic Petri dish (inner diameter 135 mm, height 15 mm). The upper loop supporting the soap film is a section of Tygon tubing (diameter 8 mm) attached to a shelf extending out from the bottom of a plastic cylinder that fits tightly around the Falcon tube and smoothly slides along it. The shelf is perforated in order to allow free circulation of air in and out of the half-catenoid region. The Petri dish was filled with a soap solution to ~ 2 mm below the top. The soap solution was a mixture of Fairy liquid detergent, glycerol and water using published proportions (recipe C in Lalli *et al.* (2023)).

Videos of the meniscus motion were captured at 200 frames/sec using a Phantom V641 high speed camera equipped with a Zeiss macro lens ($f = 60\,\mathrm{mm}$). The SPB motion was tracked by hand from those videos using Image J. Figure 3 shows the average data from two independent sessions, consisting respectively of 8 and 10 independent runs, plotted against the scaled time T from theory. At long times, the data clearly favour $q = 2/3$. The deviations visible at early times may arise from a breakdown of the quasi-static approximation when the film separates from the bath. Taking the value $q = 2/3$, there is one free parameter to compare experiment and theory, the characteristic time τ that maps the dimensional time t to the scaled time T via $T = t/\tau$, where $\tau = RA^{3/2}\mu/\gamma$. We find $\tau = 0.04\,\mathrm{s}$. Using $R = 4\,\mathrm{cm}$, $\mu = 2\,\mathrm{cP}$, and the estimated $\gamma = 25\,\mathrm{dyne/cm}$, we find $A = 11 - 18$, consistent in scale with previous observations by Cantat *et al.* (2004) and Cantat & Delannay (2005).

The agreement between theory and experiment reported provides further validation of the applicability of Bretherton's law to the motion of Plateau borders, and also a pathway forward in the quantitative description of more complex topological rearrangements of soap films (Goldstein *et al.* 2014). Chief among these are those in which the Plateau border undergoes a reconnection at the moment of singularity (Goldstein *et al.* 2010).

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