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Mapping of the relativistic kinetic balance equations onto the Klein–Gordon and second-order Dirac equations

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Abstract
In previous work we have shown that the quantum potential can be derived from the classical kinetic equations both for particles with and without spin. Here, we extend these mappings to the relativistic case. The essence of the analysis consists of Fourier transforming the momentum coordinate of the distribution function. This procedure introduces a natural parameter η with units of angular momentum. In the non-relativistic case the ansatz of either separability, or separability and additivity, imposed on the probability distribution function produces mappings onto the Schrödinger equation and the Pauli equation, respectively. The former corresponds to an irrotational flow, the latter to a fluid with non-zero vorticity. In this work we show that the relativistic mappings lead to the Klein–Gordon equation in the irrotational case and to the second-order Dirac equation in the rotational case. These mappings are irreversible; an approximate inverse is the Wigner function. Taken together, these results provide a statistical interpretation of quantum mechanics.

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1. Introduction

In recent years there has been a tremendous resurgence of interest in the hydrodynamic formulation of quantum mechanics, in large part due to its applications in semiclassical physics. This hydrodynamic representation dates back to the work of Madelung in 1926 [1] and has been extended to particles with spin, and to relativistic particles [2, 3]. The only difference between the classical Euler equations of fluid mechanics and these quantum hydrodynamic equations is the particular form of the pressure, arising from what is known as the quantum potential. A physical interpretation of the quantum potential has been lacking, primarily because it has essentially been a curiosity obtained by manipulation of the wave equations of quantum mechanics. It would seem more natural to seek a physical interpretation directly within the context of the fluid equations, for, as is well known, all fluid equations have an underlying kinetic theory. Indeed, we have recently shown [4, 5] that the hydrodynamic equations corresponding to non-relativistic quantum mechanics are no exception to this rule, implying a statistical origin of quantum mechanics.

In those previous studies we found that it is possible, depending on the requirements imposed on the Fourier transform of the one-particle probability function, to map the first balance equations obtained from the classical Boltzmann equation onto either the Schrödinger equation or the Pauli equation for a particle with spin $\frac{1}{2}$. We have also shown that the rules to obtain the hydrodynamic averages associated with the fluid equations obtained from the balance equations read like the postulates of quantum mechanics, provided the parameter η introduced in our mapping as a consequence of the Fourier transform is identified with $\hbar$. It seems natural to wonder if analogous mappings with equivalent ansatz for the one-particle probability function can be produced that lead to the Klein–Gordon and Dirac equations. As we show below, this is indeed the case. When we use the ansatz corresponding to irrotational flow it is rather straightforward to obtain the Klein–Gordon equation by following closely the steps that lead to the Schrödinger equation in the non-relativistic case. Obtaining the Dirac equation is somewhat more complex, and instead of obtaining it in standard form we are led to the second-order version of the operator. As we shall see, this outcome derives from the fact that a Lagrangian formalism is the natural framework for rotational flows. As explained previously by Feynman [6,7], the Klein–Gordon equation is easily handled within the Lagrangian formalism, but the Dirac equation is very hard to represent directly in this framework and one is always led first to its second-order version. This particular issue is also connected to the fact that it is possible to use a simple two-component spinor for the wave function as long as we work with the second-order Dirac equation. These facts, even though of great importance when finding the mappings, do not have any bearing on the results, since the full equivalence of the first- and second-order formulations of the Dirac equation has been proved by Feynman and Gell-Mann [8]. We also extend our previous non-relativistic results to shed light on the origin of the Wigner function and the coherent states.

2. The mapping

We start as usual by noting [9] that the covariant one-particle distribution function $f(x^\mu, p_\mu)$ has the properties $f \geqslant 0$, $f \to 0$ as $p_\mu \to \infty$, and the normalization

$$\int_{\Sigma_x} \int \frac{1}{c} f v^\mu \, \mathrm{d}\sigma_\mu d^4 p = \text{const}, \tag{1}$$

where Σ_x is a hypersurface in four-dimensional $\mathbf{x}$-space, with differential element $\mathrm{d}\sigma_\mu$, and

$$\frac{1}{c} v^\mu \, \mathrm{d}\sigma_\mu d^4 p \tag{2}$$

is the invariant phase space element, and $v^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, with τ being the proper time. Note that for this choice of phase space element, p_0 is considered to be an independent variable, while other representations [10] incorporate directly the constraint $p_\mu p^\mu = -mc^2$. Here, we have chosen the Kursunoglu representation for the one-particle distribution function because of its straightforward connection to the non-relativistic one, namely

$$f_{\mathrm{NR}}(\boldsymbol{x}, \boldsymbol{p})\, d^3x d^3p = \lim_{c\to\infty} \frac{v^\mu}{c} \, \mathrm{d}\sigma_\mu \, d^3p \int_{-\infty}^{\infty} f \, \mathrm{d}p_0 \tag{3}$$

and also because it is the one that leads to the standard representation of the relativistic Wigner function [11]. For simplicity of notation, we shall henceforth denote the four-vector x^μ as x and p_μ as p.

With these definitions, it is possible to derive the Boltzmann equation with a generic collision term [9]

$$p^\mu \frac{\partial f}{\partial x^\mu} + G^\mu(x, p)\frac{\partial f}{\partial p^\mu} = C(f), \tag{4}$$

where $C(f)$ is the collision integral and G^μ represents the external force, given by

$$G^\mu = \frac{e}{c} F^{\mu\nu} p_\nu, \tag{5}$$

where $F^{\mu\nu}$ is the electromagnetic tensor in the absence of dipole interaction, and

$$G^\mu = \frac{e}{c} F^{\mu\nu} p_\nu + \frac{e}{2c} N_{\alpha\beta} \partial^\mu F^{\alpha\beta}, \tag{6}$$

which corresponds to the dipole interaction, where $N^{\alpha\beta}$ is the magnetic moment tensor.

We should point out that the Liouville equation can be viewed as the particular case in which the collision integral vanishes. Here, we will make use of the crucial role the Boltzmann equation plays in fluid dynamics to find the quantum fluid equations in the relativistic case. In the standard derivation of hydrodynamics, the first two moments of the Boltzmann equation with respect to the momentum coordinate give rise to the continuity and Euler equations with an unknown pressure term, and the contribution from the collision integral vanishes due to conservation laws. To find the pressure we must determine the distribution function f, either by perturbative methods or through an ansatz. In this work we follow the analogous path in Fourier space. To calculate the Fourier space equivalent of the pressure tensor, we impose the ansatze that will lead either to the Klein–Gordon or the Dirac equations, which involve, respectively, the force laws in (5) and (6).

Let us now proceed to calculate the moments of equation (4) by averaging it with respect to p (conservation of number of particles), multiplying by p_σ and averaging over p (conservation of momentum–energy). In all cases, the right-hand side vanishes and thus the first two balance equations read

$$\int_{-\infty}^{+\infty} d^4 p p^\mu \frac{\partial f}{\partial x^\mu} = 0 \tag{7}$$

and

$$\int_{-\infty}^{+\infty} d^4 p p_\sigma \left(p^\mu \frac{\partial f}{\partial x^\mu} + G^\mu(x, p)\frac{\partial f}{\partial p^\mu} \right) = 0. \tag{8}$$

Here, we have assumed that any surface terms vanish due to the convergence properties of f. We now introduce into (7) and (8) the following representation for f,

$$f(x, p) = \frac{1}{(2\pi\eta)^4} \int_{-\infty}^{+\infty} \exp\left(-\mathrm{i}\frac{p_\mu y^\mu}{\eta}\right) \hat{f}(x, y) d^4 y \tag{9}$$

and $\hat{f}(x, y)$ is given by

$$\hat{f}(x, y) = \int_{-\infty}^{+\infty} \exp\left(\mathrm{i}\frac{p_\mu y^\mu}{\eta}\right) f(x, p) d^4 p. \tag{10}$$

With these definitions and some straightforward algebra, equations (7) and (8) become [4]

$$\lim_{y\to 0} \frac{\eta}{\mathrm{i}} \frac{\partial}{\partial x^\mu} \frac{\partial \hat{f}}{\partial y_\mu} = 0 \tag{11}$$

and

$$\lim_{y\to 0}\left[-\eta^2\frac{\partial}{\partial x^\mu}\left(\frac{\partial^2\hat{f}}{\partial y_\mu\partial y_\sigma}\right)-\frac{e\eta}{\mathrm{i}c}F^{\sigma\mu}\frac{\partial\hat{f}}{\partial y^\mu}-\frac{e}{2c}N_{\alpha\beta}\partial^\sigma F^{\alpha\beta}\hat{f}\right]=0.$$

These two limits correspond to the following averages:

$$\begin{aligned}\lim_{y\to 0}\hat{f}&=\lim_{y\to 0}\int_{-\infty}^{+\infty}\exp\left(\mathrm{i}\frac{p_\mu y^\mu}{\eta}\right)f(x,p)d^4p\\ &=\int_{-\infty}^{+\infty}f(x,p)d^4p=\frac{\rho(x)}{m},\end{aligned}\tag{12}$$

$$\begin{aligned}\lim_{y\to 0}\frac{\partial\hat{f}}{\partial y^\sigma}&=\lim_{y\to 0}\frac{\partial}{\partial y^\sigma}\int_{-\infty}^{+\infty}\exp\left(\mathrm{i}\frac{p_\mu y^\mu}{\eta}\right)f(x,p)d^4p\\ &=\frac{\mathrm{i}}{\eta}\int_{-\infty}^{+\infty}p^\sigma f(x,p)d^4p\\ &=\frac{\mathrm{i}}{\eta}\rho(x)u^\sigma(x),\end{aligned}\tag{13}$$

where we have defined the mean 4-velocity u as the average, over the momentum only, of p/m, with m being the rest mass. We can see from these expressions that $\hat{f}$ is the generating function for the averages with respect to p. Replacing these values in the balance equations we obtain the fluid equations:

$$\partial_\mu(\rho u^\mu)=0\tag{14}$$

and

$$\lim_{y\to 0}-\eta^2\frac{\partial}{\partial x^\mu}\left(\frac{\partial^2\hat{f}}{\partial y_\mu\partial y^\sigma}\right)-\frac{e}{c}\rho F_{\sigma\mu}u^\mu-\frac{e}{2mc}\rho N_{\alpha\beta}\partial^\sigma F^{\alpha\beta}=0.\tag{15}$$

The tensor in equation (15) has been evaluated in great detail for the non-relativistic case [4, 5]. Since the calculation in the relativistic case is identical, except for the use of 4-vector notation, we will only summarize the procedure. First, we introduce the canonical change of variables $y=x'-x''$ and $x=(x'+x'')/2$, which satisfies the following relationships:

$$\begin{aligned}&x'=x+\frac{y}{2},\qquad x''=x-\frac{y}{2},\\ &\frac{\partial}{\partial y}=\frac{1}{2}\left(\frac{\partial}{\partial x'}-\frac{\partial}{\partial x''}\right),\qquad\frac{\partial}{\partial x}=\left(\frac{\partial}{\partial x'}+\frac{\partial}{\partial x''}\right).\end{aligned}\tag{16}$$

Note that the limit $y\to 0$ corresponds to $x'\to x''$ and $x'=x''\equiv x$. It is at this point that we need to make some assumptions on the properties of $\hat{f}$ to continue our calculation. As is standard in the study of linear PDEs, we shall examine the possibility of separable solutions, a linear combination of which would constitute the general solution. We concentrate on two different ansatze:

(a) in the limit $x'\to x''$, the function $\hat{f}$ is fully separable in the variables x' and x'',

$$\hat{f}(x',x'')=h'(x')h''(x'')\tag{17}$$

and

(b) in the limit $x'\to x''$, the function $\hat{f}$ is the sum of two separable functions of the variables x' and x'',

$$\hat{f}(x',x'')=h'(x')h''(x'')+g'(x')g''(x'').\tag{18}$$

It is important to note that $\hat{f}$ must be real in the limit $y \to 0$: therefore, case (a), i.e. full separability, corresponds to h' and h'' being complex conjugates of each other, leaving only two independent real functions, and case (b), i.e. the sum of two separable functions, corresponds to h' and g' being complex conjugates of h'' and g'', respectively, leaving four independent real functions. Thus, adopting the ansatz of case (a) will give us enough degrees of freedom to treat irrotational flows in the context of the fluid equations (14) and (15), while adopting the ansatz of case (b) will give us enough degrees of freedom to work with vortical flows with the same equations. As has been mentioned before [4] these solutions, were they to exist, would just be a very particular subset of the solutions to the original Boltzmann equation and their importance would lie in the fact that they would be the only ones that led to the Klein–Gordon and Dirac operators, respectively. The separable solutions are of course basis functions like those that any PDE would generate. In quantum mechanics, they are known as 'pure states'. We will return to this point in the discussion of the Wigner function below.

3. The Klein–Gordon equation

In order to obtain the Klein–Gordon equation we proceed by imposing ansatz (a) on $\hat{f}$, then calculating the limit $y \to 0$ of the tensor in (15), using (5) for G^μ.

As shown previously [4], the limit $y \to 0$ with the ansatz of case (a) corresponds to

$$\lim_{y\to 0} \frac{\partial^2}{\partial y_\mu \partial y^\sigma} m \hat{f}(x', x'') = \frac{1}{4}\rho \frac{\partial^2 \ln \rho}{\partial x_\mu \partial x^\sigma} - \rho \frac{m^2}{\eta^2} u^\mu u_\sigma, \tag{19}$$

where we have defined

$$\begin{aligned} \lim_{y\to 0} \sqrt{m} h'(x') &= \psi(x), \\ \lim_{y\to 0} \sqrt{m} h''(x'') &= \psi^*(x) \end{aligned} \tag{20}$$

and the density ρ can be easily verified to be given by $\rho = \psi^* \psi$. Replacing this result in (15) with the appropriate expression for G^μ we obtain from

$$\lim_{y\to 0} \left[-\eta^2 \frac{\partial}{\partial x^\mu} \left(\frac{\partial^2 \hat{f}}{\partial y_\mu \partial y_\sigma} \right) - \frac{e\eta}{\mathrm{i}c} F^{\sigma\mu} \frac{\partial}{\partial y^\mu} \hat{f} \right] = 0 \tag{21}$$

the Euler equation that together with continuity reads [14]

$$\begin{aligned} &\partial_\mu(\rho u^\mu) = 0, \\ &m u^\mu \partial_\mu u_\sigma - \frac{\eta^2}{2m} \partial_\sigma \left(\frac{\partial_\mu \partial^\mu \rho^{1/2}}{\rho^{1/2}} \right) - \frac{e}{c} F_{\sigma\nu} u^\nu = 0, \end{aligned} \tag{22}$$

where we have reduced the second equation of this pair making use of continuity and the identity

$$\frac{1}{\rho} \partial_\mu \left(\rho \frac{\partial^2 \ln \rho}{\partial x_\mu \partial x^\sigma} \right) = 2\partial_\sigma \left(\frac{\partial_\mu \partial^\mu \rho^{1/2}}{\rho^{1/2}} \right). \tag{23}$$

Since we are working under the assumption that the Euler equations we have generated correspond to an irrotational flow, it is natural to introduce the generalized average 4-velocity $mu^\mu = \partial^\mu S + (e/c)A^\mu$. Taking advantage of the expression for the electromagnetic tensor as a function of the vector potential, $F_{\sigma\mu} = (\partial_\sigma A_\mu - \partial_\mu A_\sigma)$, and defining a new function $R = \rho^{1/2}$, the fluid equations can be rewritten as

$$2\partial_\mu R \left(\partial^\mu S + \frac{e}{c} A^\mu \right) + R \partial_\mu \left(\partial^\mu S + \frac{e}{c} A^\mu \right) = 0 \tag{24}$$

and

$$R\left(\partial^{\mu}S+\frac{e}{c}A^{\mu}\right)\left(\partial_{\mu}S+\frac{e}{c}A_{\mu}\right)-\eta^{2}\partial_{\mu}\partial^{\mu}R+Km^{2}=0, \tag{25}$$

where we have a yet undetermined constant K as a consequence of having integrated the second equation once. Now, multiplying the first equation by (i/η) and the second one by $-1/\eta^2$ and adding them it is easy to see that we can use the standard Hopf–Cole transformation $\Omega = \ln R + (\mathrm{i}/\eta)S = \ln\Psi$ to rewrite the pair of equations (24) and (25) as

$$\left[\left(\mathrm{i}\eta\partial_{\mu}-\frac{e}{c}A_{\mu}\right)^{2}-Km^{2}\right]\Psi=0. \tag{26}$$

It is apparent that the choices $K = c^2$ and $\eta = \hbar$ would make (26) the Klein–Gordon equation.

Note that the solutions generated by (26) are only guaranteed to hold in the limit $y \to 0$. If we ignore the constraint of the limit and simply assume that the Fourier-transformed distribution function remains separable for *all* y, then the original distribution function would be obtained by the inverse transform,

$$F_{\mathrm{W}}(x,p)=\left(\frac{1}{2\pi\eta}\right)^{4}\int_{-\infty}^{\infty}\Psi^{\dagger}\left(x+\frac{y}{2}\right)\Psi\left(x-\frac{y}{2}\right)\exp\left(-\mathrm{i}\frac{p_{\mu}y^{\mu}}{\eta}\right)d^{4}y, \tag{27}$$

which is nothing other than the relativistic scalar generalization of the standard Wigner function [11].

Such Wigner functions are not proper probability distributions because they are not positive definite (although there have been some attempts to circumvent this problem in the relativistic case [12]). In the present framework, the pure states also do not necessarily satisfy the minimum uncertainty condition $\Delta x\Delta p = \eta/2$, which is a direct consequence of the Fourier transform applied to the distribution function. If, instead, we construct a linear combination of these separable solutions that at all times obeys the minimum-uncertainty condition, then the anti-transform does yield an acceptable distribution function. These combinations are, of course, the coherent states. A well-known non-relativistic example of this is provided by the harmonic oscillator, whose coherent state distribution function is not only positive definite, but also satisfies the Boltzmann equation.

4. The Dirac equation

Here, we consider the case of vortical flows. As mentioned above, the outcome of this mapping leads to the second-order Dirac equation. In the non-relativistic case, vortical flows can only be treated in all generality by using a Lagrangian formalism [5]. This is also true in the relativistic case. Thus, following closely the steps for the non-relativistic calculation, the relativistic mapping begins by invoking ansatz (b) for our function $\hat{f}$. Then, we evaluate the tensor in the momentum balance equation including the dipole interaction term, i.e. using the expression (6) for the external force, and recast the whole expression as the Euler fluid equations. Next, we introduce the action corresponding to the fluid equations and perform the variation to prove that this Lagrangian density indeed corresponds to the equations of motion. Finally, we introduce a change of variables and when the variation is performed on the new variables the resulting equation of motion is indeed the second-order Dirac equation when η is set equal to $\hbar$.

Under ansatz (b), the tensor of (15) in the limit $y \to 0$ has the value [5]

$$\begin{aligned}\lim_{y\to 0}\frac{\partial}{\partial y^{\mu}}\frac{\partial}{\partial y_{\sigma}}m\hat{f}(x',x'')&=\lim_{x',x''\to x}\frac{1}{4}\left(\frac{\partial}{\partial x'^{\mu}}-\frac{\partial}{\partial x''^{\mu}}\right)\left(\frac{\partial}{\partial x'_{\sigma}}-\frac{\partial}{\partial x''_{\sigma}}\right)m[h'h''+g'g'']\\&=\frac{1}{4}\left[\rho\frac{\partial^{2}\ln\rho}{\partial x^{\mu}\partial x_{\sigma}}-4\rho\frac{m^{2}}{\eta^{2}}u_{\mu}u^{\sigma}-\rho\frac{\partial\Sigma_{i}}{\partial x^{\mu}}\frac{\partial\Sigma_{\mathrm{i}}}{\partial x_{\sigma}}\right],\end{aligned} \tag{28}$$

where we have defined

$$\begin{aligned}
&\lim_{y\to 0}\sqrt{m}h'(x') = \psi_1(x),\\
&\lim_{y\to 0}\sqrt{m}h''(x'') = \psi_1^*(x),\\
&\lim_{y\to 0}\sqrt{m}g'(x') = \psi_2(x),\\
&\lim_{y\to 0}\sqrt{m}g''(x'') = \psi_2^*(x)
\end{aligned} \tag{29}$$

introduced the notation

$$\Sigma_i = \frac{\psi^\dagger\sigma_i\psi}{\psi^\dagger\psi}, \tag{30}$$

where

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \psi^\dagger = (\psi_1^*, \psi_2^*) \tag{31}$$

and σ_i are the Pauli matrices, and then used (12) and (13) for the density ρ and the mean 4-velocity u_μ, which can be rewritten as a function of ψ as $\rho = \psi^\dagger\psi$ and $u^\mu = -(\eta/2m\mathrm{i})(\psi^\dagger\partial_\mu\psi - \psi\partial_\mu\psi^\dagger)$. Expression (28), even though formally correct, is not very useful in its present form since Σ, as presented in (30), reads like a 3-vector (note that all three components Σ_i are real). This problem can be easily solved by introducing the 4-spinor

$$\Psi = \begin{pmatrix}\psi\\ -\psi\end{pmatrix}, \qquad \Psi^\dagger = (\psi^\dagger, -\psi^\dagger). \tag{32}$$

These definitions allow us also to rewrite (28) in a much more convenient and instructive way:

$$\lim_{y\to 0}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y_\sigma}m\hat{f}(x',x'') = \frac{1}{4}\left[\rho\frac{\partial^2\ln\rho}{\partial x^\mu\partial x_\sigma} - 4\rho\frac{m^2}{\eta^2}u_\mu u^\sigma - \frac{1}{4}\rho\frac{\partial M_{\alpha\beta}}{\partial x^\mu}\frac{\partial M^*_{\alpha\beta}}{\partial x_\sigma}\right], \tag{33}$$

where $\rho = (1/2)\Psi^\dagger\Psi$ and $u_\mu = -(\eta/2m\mathrm{i})(1/2)(\Psi^\dagger\partial_\mu\Psi - \Psi\partial_\mu\Psi^\dagger)$. The tensor $M_{\alpha\beta}$ is defined as

$$M_{\alpha\alpha} = 0, \qquad M_{\alpha\beta} = \frac{1}{2}\mathrm{i}\frac{\Psi^\dagger[\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha]\Psi}{\Psi^\dagger\Psi}, \tag{34}$$

where the γ_μ are the gamma matrices,

$$\gamma_t = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad \gamma_i = \begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}. \tag{35}$$

The explicit expression for the tensor M is then

$$M = \begin{pmatrix}0 & \Sigma_z & -\Sigma_y & \mathrm{i}\Sigma_x\\ -\Sigma_z & 0 & \Sigma_x & \mathrm{i}\Sigma_y\\ \Sigma_y & -\Sigma_x & 0 & \mathrm{i}\Sigma_z\\ -\mathrm{i}\Sigma_x & -\mathrm{i}\Sigma_y & -\mathrm{i}\Sigma_z & 0\end{pmatrix}. \tag{36}$$

Introducing these definitions into equations (14) and (15), with the force G_μ given by (6), we obtain after some lengthy, but straightforward, algebra the following fluid equations

$$\begin{aligned}
&\partial_\mu(\rho u^\mu) = 0,\\
&m u^\mu\partial_\mu u_\sigma - \frac{\eta^2}{2m}\partial_\sigma\left(\frac{\partial_\mu\partial^\mu\rho^{1/2}}{\rho^{1/2}}\right) + \frac{\eta^2}{4m\rho}\partial_\mu\left(\frac{\rho}{4}\frac{\partial M_{\alpha\beta}}{\partial x^\mu}\frac{\partial M^*_{\alpha\beta}}{\partial x_\sigma}\right) - \frac{e}{c}F_{\sigma\nu}u^\nu\\
&\qquad - \frac{e\eta}{4mc}M_{\alpha\beta}\partial_\sigma F_{\alpha\beta} = 0,
\end{aligned} \tag{37}$$

where once again we have used identity (23) and continuity and identified the tensor M with the magnetic moment tensor through the relationship $N = (\eta/2)M$. The motivation to make this identification between M and N lies in the fact that both quantities transform in the same manner and have the exact number of powers of $\eta/2$ to ascribe to them units of angular momentum. Moreover, from equation (36) we see that there are only three non-zero elements in M that are different from each other. Since these three elements are equal to the non-relativistic components of the vector angular momentum in three dimensions it is natural to think of $(\eta/2)M$ (and also N) as the relativistic tensor that corresponds to the axial 3-vector $\boldsymbol{\Sigma}$. As we have shown before [5], it is possible to express these quantities in a more physical context by introducing the Clebsch variables ζ and ω [15]. As a consequence of representing $\boldsymbol{\Sigma}$ with the Pauli matrices, ζ corresponds to the z-component of the vector $\boldsymbol{\Sigma}$ and ω corresponds to the azimuthal angle, i.e. the canonical conjugate variable of Σ_z. Then, as a function of the components of $\boldsymbol{\Sigma}$ (or equivalently the elements of the tensor M), the expressions for ζ and ω are

$$\zeta = \frac{\eta}{2}\Sigma_z, \qquad \omega = \tan^{-1}\left(\frac{\Sigma_x}{\Sigma_y}\right). \tag{38}$$

If we also introduce the angle θ to represent the angle that Σ makes with the z-axis the three distinct elements of M can be expressed as

$$\begin{aligned} \Sigma_x &= \sin\theta\sin\omega, \\ \Sigma_y &= \sin\theta\cos\omega, \\ \Sigma_z &= \cos\theta. \end{aligned} \tag{39}$$

We can now rewrite the last term in equation (33) as a function of the Clebsch variables. After some algebra, it is easy to verify that

$$\frac{\partial M_{\alpha\beta}}{\partial x^\mu}\frac{\partial M^*_{\alpha\beta}}{\partial x_\sigma} = \frac{\partial_\mu\zeta\,\partial^\sigma\zeta}{q} + \frac{4}{\eta^2}q\,\partial_\mu\omega\partial^\sigma\omega, \tag{40}$$

where we have defined q as

$$q(\zeta) \equiv q = \frac{\eta^2}{4} - \zeta^2, \tag{41}$$

a function of ζ only.

Now, we show that the action that corresponds to the equations of motion (37) is

$$\begin{aligned} \mathcal{A} = -\int d^4x \Bigg[&\frac{\rho}{2m}\left(\partial_\mu S + \zeta\,\partial_\mu\omega + \frac{e}{c}A_\mu\right)^2 - mc^2\rho + \frac{\eta^2}{8m}\frac{\partial_\mu\rho\partial^\mu\rho}{\rho} + \frac{\eta^2}{8m} \\ &\times\left(\frac{(\partial_\mu\zeta)^2}{q} + \frac{4}{\eta^2}q(\partial_\mu\omega)^2\right) - \frac{e\eta}{4mc}\rho M_{\alpha\beta}F^{\alpha\beta}\Bigg], \end{aligned} \tag{42}$$

where we have already substituted into (42) the expression $u_\mu = 1/m(\partial_\mu S + \zeta\,\partial_\mu\omega + (e/c)A_\mu)$, which is the result of the variation of $\mathcal{A}$ with respect to u^μ, i.e. $(\delta\mathcal{A}/\delta u^\mu)$. The remaining variations are given by

$$\begin{aligned} &\frac{\delta\mathcal{A}}{\delta\omega} : u^\mu\partial_\mu\zeta + \frac{1}{m}\partial_\mu\left(\rho q\,\partial^\mu\omega\right) + \frac{e\eta}{4mc}\frac{\partial M_{\alpha\beta}}{\partial\omega}F_{\alpha\beta} = 0, \\ &\frac{\delta\mathcal{A}}{\delta\zeta} : u^\mu\partial_\mu\omega - \frac{\eta^2}{8m}\frac{q'}{q}(\partial_\mu\zeta)^2 + \frac{1}{2m}q'(\partial_\mu\omega)^2 - \frac{\eta^2}{4m}\partial_\mu\left(\rho\frac{\partial_\mu\zeta}{q}\right) - \frac{e\eta}{4mc}\frac{\partial M_{\alpha\beta}}{\partial\zeta}F_{\alpha\beta} = 0, \\ &\frac{\delta\mathcal{A}}{\delta S} : \partial_\mu\left[\rho\frac{1}{m}\left(\partial^\mu S + \zeta\,\partial^\mu\omega + \frac{e}{c}A^\mu\right)\right] \equiv \partial_\mu(\rho u^\mu) = 0, \\ &\frac{\delta\mathcal{A}}{\delta\rho} : \frac{1}{2m}\left(\partial_\mu S + \zeta\,\partial_\mu\omega + \frac{e}{c}A_\mu\right)^2 - \frac{\eta^2}{2m}\frac{\partial^\mu\partial_\mu\rho^{1/2}}{\rho^{1/2}} - mc^2 + \frac{\eta^2}{8m}\left(\frac{(\partial_\mu\zeta)^2}{q} + \frac{4}{\eta^2}q(\partial_\mu\omega)^2\right) \\ &\qquad - \frac{e\eta}{4mc}M_{\alpha\beta}F^{\alpha\beta} = 0. \end{aligned} \tag{43}$$

Now, we take the derivative $\partial_\sigma(\delta\mathcal{A}/\delta\rho) = 0$ and substitute $u_\mu = 1/m(\partial_\mu S + \zeta\partial_\mu\omega - (e/c)A_\mu)$. Then, we use continuity and replace the values of the variations with respect to ω and ζ to finally obtain (37). Note also that continuity is simply given by the variation of $\mathcal{A}$ with respect to S, $(\delta\mathcal{A}/\delta S) = 0$.

Finally, to show that the action $\mathcal{A}$ also gives rise to the second-order Dirac equation we introduce the following expression for the 4-spinor Ψ [16]:

$$\Psi = \begin{pmatrix} \psi \\ -\psi \end{pmatrix} \qquad \text{with } \psi = R\mathrm{e}^{\mathrm{i}S/\eta}\begin{pmatrix} \cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\omega/2} \\ \mathrm{i}\sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\omega/2} \end{pmatrix}. \tag{44}$$

In this representation, the action (42) becomes

$$\mathcal{A}_\Psi = -\frac{1}{2}\int d^4x\left[\frac{1}{2m}\left(\mathrm{i}\eta\frac{\partial\Psi^\dagger}{\partial x^\mu} + \frac{e}{c}\Psi^\dagger A_\mu\right)\left(-\mathrm{i}\eta\frac{\partial\Psi}{\partial x_\mu} + \frac{e}{c}A^\mu\Psi\right) - mc^2\Psi^\dagger\Psi \right.$$
$$\left. -\frac{e\eta}{2mc}\left(\frac{1}{2}M_{\alpha\beta}F^{\alpha\beta}\right)\Psi^\dagger\Psi\right]. \tag{45}$$

Then, taking the variation with respect to $\Psi^\dagger$ [16] we obtain

$$\left[\left(-\mathrm{i}\eta\frac{\partial}{\partial x^\mu} + \frac{e}{c}A_\mu\right)^2 - m^2c^2 - \frac{e\eta}{2c}\sigma_{\alpha\beta}F^{\alpha\beta}\right]\Psi = 0, \tag{46}$$

where $\sigma_{\alpha\beta} = (\mathrm{i}/2)(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$. As with the Klein–Gordon equation, if we choose to identify $\eta = \hbar$, then equation (46) reads

$$\left[\left(\mathrm{i}\hbar\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu\right)^2 - \frac{e\eta}{2c}\sigma_{\alpha\beta}F^{\alpha\beta}\right]\Psi = m^2c^2\Psi, \tag{47}$$

which is the second-order Dirac equation in the Feynman–Gell-Mann formulation [8].

5. Conclusions

In this work we have performed the relativistic extension of previous results connecting the Boltzmann equation to the Schrödinger and Pauli operators. We have found that the Fourier transform of the one-particle distribution function of the classic relativistic Boltzmann equation with respect to the momentum variable can be mapped either onto the Klein–Gordon or the Dirac equations. As in the non-relativistic case, the first part of the mapping leads to a set of Euler equations for a compressible fluid. From them, the analysis of irrotational flows coupled to the ansatz of separability applied to the one-particle probability function leads to the Klein–Gordon equation for particles with no spin. A similar analysis for rotational flows and the ansatz of separability and addition leads to the Dirac equation for particles with spin $\eta/2$. The rules to calculate the averages of physical quantities in the p-conjugate space are the four vector versions of the rules found in the non-relativistic case, which read like the postulates of quantum mechanics with η replaced by $\hbar$ [4]. In addition, by observing that pure states generally violate the constraints of the Fourier transform, we have provided an explanation for the shortcomings of their Wigner distribution and how imposing Fourier constraints is equivalent to the requirement of minimum uncertainty in quantum mechanics, both leading to coherent states.

There is a very interesting consequence to the ansatze (a) and (b) imposed on $\hat{f}$. As emphasized previously, these solutions form a very small subset of all possible solutions to the balance equations. Returning to the original Boltzmann equation, we see that its right-hand side is the collision integral that includes a first approximation to the two-particle probability function $f_2(x_1, p_1, x_2, p_2)$ constructed as a combination of products of one-particle probability functions. Independent of the approximation used, f_2 must be symmetric under the exchange of coordinates $1 \leftrightarrow 2$, so $f_2(x_1, p_1, x_2, p_2) = f_2(x_2, p_2, x_1, p_1)$ [13]. When the approximation for f_2 used in the Boltzmann equation is adopted this symmetry will also hold true for its Fourier transform $\hat{f}_2(x_1, y_1, x_2, y_2)$. If we now invoke the separability condition $\hat{f}_2 = \Psi^\dagger(x_1'', x_2'')\Psi(x_1', x_2')$ [4], and note that we are working with identical particles, the functions Ψ must be either symmetric or antisymmetric under the exchange of variables $1 \leftrightarrow 2$ so that $\hat{f}_2$ will be symmetric. It may be possible to prove that the case which maps onto the Klein–Gordon equation requires Ψ to be symmetric, while the Dirac case requires antisymmetry. Such a proof would need a detailed study of the Bogoliubov hypothesis that leads to the Boltzmann equation.

One last issue that we would like to mention relates to the Proca equations that govern particles of spin 1 or higher. Since these can be developed from the Dirac equation [17] it seems reasonable to think that there might be a new ansatz for $\hat{f}$ that would lead to the equations for higher spin. Unfortunately, we have not yet been able to find a satisfactory derivation that would settle the issue either way. Perhaps these issues should be the subject of further work.

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