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The Korteweg–de Vries Hierarchy as Dynamics of Closed Curves in the Plane

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The Korteweg–de Vries, modified Korteweg–de Vries, and Harry Dym hierarchies of integrable systems are shown to be equivalent to a hierarchy of chiral shape dynamics of closed curves in the plane. These purely local dynamics conserve an infinite number of global geometric properties of the curves, such as perimeter and enclosed area. Several techniques used to study these integrable systems are shown to have simple differential-geometric interpretations. Connections with incompressible, inviscid fluid flow in two dimensions are suggested.

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In describing the dynamics of shapes in many physical and biological systems, one encounters the mathematical problem of preserving *global* geometric quantities such as surface area and enclosed volume. Examples of such dynamics include the motion of regions of incompressible fluids (conserving volume), of polymers (conserving length and topological quantities), and of cell membranes (conserving both enclosed volume and surface area). For systems such as membranes and polymers, a representation of the dynamics in terms of the motion of *surfaces* of one or two dimensions is clearly most natural; in the fluid dynamical context it is often conceptually simpler to focus on the boundary of the fluid rather than the bulk, again leading to surface dynamics. Within this description, the imposition of global constraints generally entails strong nonlocality as a consequence of long-range hydrodynamic interactions [1,2], which may enter the surface dynamics through Lagrange multipliers conjugate to the conserved quantities [3,4].

Here, we study examples of surface dynamics that are purely *local*, yet nevertheless maintain global constraints. Focusing on the simplest case of closed curves in the plane, and appealing to very general considerations regarding the conservation of perimeter and enclosed area, we find a class of motions that in fact conserves an *infinite* number of global geometric quantities. These dynamics are closely related to the hierarchies of integrable systems of the Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) types.

Our main results may be summarized as follows. Each equation of the hierarchies is associated with a particular choice of curve dynamics of the form

$$\mathbf{r}_t = U\hat{\mathbf{n}} + W\hat{\mathbf{t}}\,, \tag{1}$$

where $\hat{\mathbf{n}}(s,t)$ and $\hat{\mathbf{t}}(s,t)$ are the unit normal and tangent vectors at arclength s and time t at the point $\mathbf{r}(s,t)$, and a subscript indicates differentiation. We restrict our attention to purely local normal and tangential velocities $U = U(\kappa, \kappa_s, \ldots)$ and $W = W(\kappa, \kappa_s, \ldots)$, where κ is the curvature. These dynamics thus bear a strong resemblance to "geometric" models of interface evolution proposed in the study of crystal growth [5]. By imposing the condition that arclength and time derivatives commute, we find that W is determined by U. After recasting the dynamics (1) for the vector $\mathbf{r}(s,t)$ into an evolution equation for the curvature, we find that for a particular sequence of functions U, the curvature equations coincide with the evolution equations of the mKdV hierarchy. The functions U obey a recursion relation, discovered earlier in a different context [6]. These velocity functions are, in turn, the Hamiltonian formulation of the mKdV dynamics.

The mKdV hierarchy is related to the KdV hierarchy through the *Miura transformation* [7], which, in the present context, provides a link between the dynamics of the curvature and that of the curve itself when viewed in the complex plane. Indeed, the appearance of a Schrödinger-like operator in the inverse scattering transform method [8] and in the Lax formulation [9] are naturally associated with the properties of the curve in the complex plane. We remark also on the relationship between the mKdV and "Harry Dym" (HD) hierarchies [10,11], demonstrating that the HD hierarchy corresponds to a Eulerian rather than Lagrangian view of the interface dynamics. We thus conclude that the KdV, mKdV, and HD hierarchies are three equivalent views of the same underlying shape dynamics. These considerations suggest a strong link between KdV dynamics and those of two-dimensional inviscid, incompressible fluids [12]. The recent appearance of the HD equation in studies of the Saffman-Taylor problem [13] lend support to this hypothesis.

Connections between the differential geometry of curve motion and integrable systems have been noted before. One example [14] is that the nonlinear Schrödinger equation describes the dynamics of a thin, nonstretching vortex filament, while a second [15] extends that analysis to include more general types of motion and shows how they are connected to other integrable systems such as the sine-Gordon equation. For the special case of a space curve with constant torsion, a relation to one member of the mKdV hierarchy was found. On a more formal level, there have also been connections between the mathematics of these integrable systems and the Lie algebras associated with curves [16], as well as the theory of differential forms [17].

We begin by establishing notation and considering general features of the motion of a curve in the plane [4,5]. The unit tangent vector in (1) satisfies $\hat{\mathbf{t}}=\mathbf{r}_s$, and is related to the curvature and normal vector by $\hat{\mathbf{t}}_s=-\kappa\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_s=\kappa\hat{\mathbf{t}}$. With the dynamics of the curve as in (1), and the length and area of the curve given by $L=\int ds$, $A=\frac{1}{2}\int ds\,\mathbf{r}\times\mathbf{r}_s$, we obtain the time derivatives

$$L_t=\oint ds(\kappa U+W_s)=\oint ds\,\kappa U \tag{2}$$

and

$$A_t=\oint ds\,U\,, \tag{3}$$

where in the second relation of (2) we have assumed that W is periodic. From these, we see that *global* length and area conservation constrain only the normal velocity U, leaving W unspecified. One may choose to conserve arclength *locally* [4] by demanding that the integrand in (2) vanish identically for all s, yielding a differential equation for W,

$$W_s=-\kappa U\,.$$

Integrating this relation determines W up to an arbitrary function of time $c(t)$; $W(s)=-\int^s ds'\kappa U+c\equiv-\partial^{-1}\kappa U$. This choice implies that arclength and time derivatives commute,

$$[\partial_s,\partial_t]\mathbf{r}=(W_s+\kappa U)\hat{\mathbf{t}}=0\,.$$

In terms of an arbitrary parametrization α, the arclength s is $s(\alpha,t)=\int^\alpha d\alpha'\sqrt{g}$, where the metric $g=\mathbf{r}_\alpha\cdot\mathbf{r}_\alpha$. The condition $W_s=-\kappa U$ then implies $g_t=0$, so the parametrization does not evolve in time: $s_t(\alpha,t)=0$.

Note that any dynamics in the form (1), with W determined as above, may be recast as an equation for the curvature [5],

$$\kappa_t=-\Omega U,\quad \Omega=\partial_{ss}+\kappa^2+\kappa_s\partial^{-1}\kappa\,. \tag{4}$$

The mKdV hierarchy.—One notices from (2) and (3) that area and perimeter are conserved if U and κU are total derivatives with respect to arclength of any periodic functions. The simplest pair of this type is

$$U^{(1)}=0,\quad W^{(1)}=-1\,, \tag{5}$$

where we have set $c\equiv-1$. The curvature then evolves according to

$$\kappa_t=-\kappa_s\,, \tag{6}$$

a reparametrization of the curve. The less trivial choice [15],

$$U^{(2)}=\kappa_s,\quad W^{(2)}=-\tfrac{1}{2}\kappa^2\,, \tag{7}$$

with $c\equiv0$ yields

$$\kappa_t=-\kappa_{sss}-\tfrac{3}{2}\kappa^2\kappa_s\,, \tag{8}$$

which is the modified Korteweg-de Vries equation [18].

Equations (6) and (8) are in fact the first two members of the mKdV hierarchy of integrable systems. The next member,

$$\kappa_t=-\kappa_{5s}-\tfrac{15}{8}\kappa^4\kappa_s-\tfrac{5}{2}\kappa_s^3-\tfrac{5}{2}\kappa^2\kappa_{sss}-10\kappa\kappa_s\kappa_{ss}\,, \tag{9}$$

corresponds to the choice

$$U^{(3)}=\kappa_{sss}+\tfrac{3}{2}\kappa^2\kappa_s\,, \tag{10}$$

$$W^{(3)}=-\tfrac{3}{8}\kappa^4+\tfrac{1}{2}\kappa_s^2-\kappa\kappa_{ss}\,. \tag{11}$$

From these results, it is clear that the right-hand side of each curvature evolution equation defines the normal velocity U of the succeeding equation. Thus, $\kappa_t=-\Omega U^{(n)}$, with

$$U^{(n)}=\Omega U^{(n-1)}=\Omega^{n-2}\kappa_s\quad(n\geq2)\,, \tag{12}$$

and $W^{(n)}=-\partial^{-1}\kappa U^{(n)}$. Note that each of the functions $U^{(n)}$ is *chiral*, i.e., odd under the transformation $s\to-s$. The recursion relation connecting the successive members of the mKdV hierarchy was known from earlier work of Chern and Peng [6]. The operator Ω is seen from (4) to have universal geometrical significance in defining the curvature evolution under arclength-conserving dynamics.

In addition to conserving length and area, by construction, these dynamics also possess an infinite number of constants of motion of the mKdV hierarchy. In the language of curve motion, these are integrals of polynomials in the curvature and its derivatives. Each of these conserved quantities $H_k = \oint ds\, h_k$ has a density h_k obeying a continuity equation

$$\partial_t h_k + \partial_s j_k = 0 ,$$

where the j_k are the associated current densities. For mKdV, which is already in the form of a continuity equation, the successive conserved densities are

$$h_2 = -\tfrac{1}{2}\kappa^2, \quad h_3 = -\tfrac{3}{8}\kappa^4 + \tfrac{1}{2}\kappa_s^2 - \kappa\kappa_{ss} , \tag{13}$$

with $j_2 = -\frac{3}{8}\kappa^4 - \kappa\kappa_{ss} + \frac{1}{2}\kappa_s^2$, etc. Note that the h_k's are the successive tangential velocities of the hierarchy.

Quantities such as the area enclosed by a curve are not directly expressible in terms of the curvature and its derivatives, but rather correspond to multiple *integrals* of the curvature. The existence of conserved quantities of this type may be related to the "prolongation structures" discussed by Wahlquist and Estabrook [17].

It is worth noting that other choices of velocities, such as $U = \kappa^n \kappa_s$ and $W = -\kappa^{n+2}/(n+2)$, also conserve perimeter and enclosed area, but it is not known if the resulting dynamics is integrable.

The KdV hierarchy.— The KdV hierarchy for a variable $u(s,t)$ can be cast in a convenient form in terms of the nth-order conserved densities $T^{(n)}$ of the KdV equation [19,20] $u_t + u_{sss} - 3uu_s = 0$, which parallel those of the mKdV hierarchy:

$$u_t + \partial_s K^{(n)} = 0 , \tag{14}$$

where the current density $K^{(n)} = \delta T^{(n)}/\delta u$ with $T^{(1)} = u$, $T^{(2)} = \frac{1}{2}u^2$, etc. The two hierarchies are connected by the Miura transformation [7]

$$u = -\tfrac{1}{2}\kappa^2 - i\kappa_s . \tag{15}$$

Apart from overall rescaling and terms that are total derivatives, the conserved densities of mKdV [Eq. (13)] are the Miura-transformed $T^{(n)}$ of KdV. With the general equation of motion for the curvature, we find by substitution of (15) into (14) that every order [6] of the KdV hierarchy factorizes into

$$(\partial_s - i\kappa)(\kappa_t + \Omega U^{(n)}) = 0 ,$$

so that if $\kappa(s,t)$ satisfies the nth-order mKdV equation, then u satisfies the nth-order KdV equation. The factor $\partial_s - i\kappa$ may be interpreted as a covariant derivative.

One observes that for the KdV equation $W^{(2)} - iU^{(2)} = u$ under the Miura transformation. The significance of the combination $W - iU$ is best seen in a representation in which a point on the curve is given by $z(s,t) = x(s,t) + iy(s,t)$, with tangent and normal vectors

$$\hat{\mathbf{t}} \to e^{i\theta}, \quad \hat{\mathbf{n}} \to -ie^{i\theta} ,$$

$\theta(s,t)$ being the tangent angle. The dynamics of the curve (1) are then

$$z_t = (W^{(n)} - iU^{(n)})e^{i\theta} . \tag{16}$$

Under the Miura transformation (15) the general result connects the complex velocity and the current density,

$$W^{(n)} - iU^{(n)} = K^{(n)} \quad (u = -\tfrac{1}{2}\kappa^2 - i\kappa_s) . \tag{17}$$

For the KdV equation, the Galilean invariance under the transformations $s \to s + 3\lambda t$, $u \to u + \lambda$, for any real constant λ, is seen to be equivalent to a redefinition of the tangential velocity $W \to W + \lambda$ through the correspondence $u = W - iU$. Given the tangent angle $\theta(s)$, one may also describe the *mirror image* of the curve through the duals of the tangent and normal vectors $\hat{\mathbf{t}}^* = \exp[-i\theta(s)]$ and $\hat{\mathbf{n}}^* = i\hat{\mathbf{t}}^*$. The velocity (16) of the mirror image is then $(W + iU)e^{-i\theta}$. The original and dual descriptions, when combined with Galilean invariance, then define two different Miura transformations

$$u_\pm = -\tfrac{1}{2}\kappa^2 \pm i\kappa_s + \lambda ,$$

a well-known Bäcklund pair [20] for the KdV equation.

The Lax formalism.— The Riccati equation connecting u and κ under the Miura transformation is linearized by the substitution $\kappa = 2i\psi_s/\psi$, yielding a Schrödinger equation for ψ,

$$(2\partial_{ss} + u)\psi = 0 . \tag{18}$$

This and the relation $\kappa = \theta_s$ implies that the "wave function" $\psi = \exp(-i\theta/2)$, i.e., the square root of the dual tangent vector.

The operator $L = 2\partial_{ss} + u$ in (18) plays a central role in the Lax description of the KdV hierarchies, in which the KdV equations themselves are cast in the form $L_t + [L, M^{(n)}] = 0$, where the operators $M^{(n)}$ are such that $[L, M^{(n)}]$ is multiplicative, a condition connected with the constancy of the eigenvalues of L [20]. As a consequence, ψ evolves in time as

$$\psi_t = M^{(n)}\psi , \tag{19}$$

and using the explicit forms of M [20], this is readily seen to generate the time evolutions of the tangent angle θ,

$$\theta_t = -\theta_s, \quad \theta_t = -\theta_{sss} - \tfrac{1}{2}\theta_s^2 ,$$

etc. By differentiation with respect to arclength, these are equivalent to the curvature evolutions in Eqs. (6), (8), and so on.

Eulerian description.— An alternative to the *Lagrangian* description of curve motion in which points on the curve are labeled by arclength s is the *Eulerian* labeling by position in space $z(s)$, determined by the nonlocal relation

$$z(s,t) = \int^s ds' e^{i\theta(s',t)} . \tag{20}$$

Geometric quantities may then be defined in terms of z as $\theta = -i\ln(z_s)$, $\kappa = -iz_{ss}/z_s$, etc. The dynamics of the tangent vector then have the form $\partial_t z_s = \partial_s[(W - iU)z_s]$. If we now consider [10,11,21] redefining the independent variable to be $z(s,t)$ and define the new function

$Y(z(s,t),t) = \exp[i\theta(z(s,t),t)]$, then

$$Y_t = Y^2 \partial_z (W - iU), \qquad (21)$$

where W and U are viewed as functions of $Y(z,t)$ through $\kappa = -iY_z$, etc., and $\partial_z = Y\partial_s$.

From the above, it is clear that we may associate an equation of the form (21) with each member of the mKdV hierarchy. For example, the simplest dynamics $W = -1$, $U = 0$ yields $Y_t = 0$, illustrating that reparametrizations do not involve actual motions of the curve in space. For the next members of the hierarchy we obtain the Harry Dym equation,

$$Y_t = -Y^3 Y_{zzz}, \qquad (22)$$

and an equation previously found [11] to be transformable to Eq. (9),

$$Y_t = -Y^5 Y_{5z} - 5Y^4 Y_z Y_{4z} - 5Y^4 Y_{zz} Y_{zzz} - \tfrac{5}{2} Y^3 Y_z^2 Y_{zzz},$$

and so on.

Relevance to incompressible hydrodynamic flows. —The conservation of enclosed area for each of the curve dynamics associated with the mKdV hierarchy suggests that they may describe the motion of boundaries of incompressible fluids. Recall that the Kelvin circulation theorem [22]

$$\oint_{\mathcal{C}} d\mathbf{l} \cdot \mathbf{v} = \text{const}, \qquad (23)$$

where $\mathcal{C}$ is a contour in the fluid flow and $\mathbf{v}$ is the local velocity, follows as a consequence of the Euler equation of incompressible inviscid fluid flow. Its validity here is a direct consequence of the conservation laws for the tangential velocities in Eq. (13). This may be a concrete realization of observations [12,23] concerning the preservation of *coadjoint orbits* of certain dual Lie algebras observed with incompressible fluid flow. Thus, it may be conjectured that to each member of the hierarchy there is an associated two-dimensional flow field, but the construction of these flows is an as yet unsolved problem.

In summary, we have shown that the mathematics of the KdV hierarchy finds a natural interpretation in the language of differential geometry as area- and perimeter-preserving dynamics of plane curves. We may naturally ask the question: Do these geometric considerations suggest a method of constructing integrable systems in higher dimensions?

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