6 Subgradient method

In this lecture we look at the problem of minimizing a general nonsmooth convex function $f(x)$.

Subgradient method The subgradient method to minimize $f(x)$ works as follows. Choose $x_0 \in \mathbb{R}^n$ and iterate, for $k \geq 0$:

$$x_{k+1} = x_k - t_k g_k$$

where $g_k \in \partial f(x_k)$ is a subgradient of f at x_k and $t_k > 0$ is the step size.

Note: A negative subgradient is not necessarily a descent direction, i.e., it is possible that $f(x - tg) > f(x)$ for all $t > 0$ (small enough). For example $f(x) = |x|$, $x = 0$ and $g = -1 \in \partial f(0)$.

Convergence analysis of subgradient method:

$$\begin{aligned} \|x_{k+1} - x^*\|_2^2 &= \|x_k - t_k g_k - x^*\|_2^2 \\ &= \|x_k - x^*\|_2^2 - 2t_k g_k^T(x_k - x^*) + t_k^2\|g_k\|_2^2 \\ &\leq \|x_k - x^*\|_2^2 + t_k^2\|g_k\|_2^2 + 2t_k(f^* - f(x_k)) \end{aligned} \tag{1}$$

where in the last line we used the fact that $g_k \in \partial f(x_k)$. Applying this inequality recursively to $\|x_k - x^*\|_2^2$, we get at the end:

$$\|x_{k+1} - x^*\|_2^2 \;\leq\; \|x_0 - x^*\|_2^2 + \sum_{i=0}^{k} t_i^2\|g_i\|_2^2 + 2\sum_{i=0}^{k} t_i(f^* - f(x_i)) \tag{2}$$

which after rearranging, and using $\|x_{k+1} - x^*\|_2^2 \geq 0$, gives us

$$\sum_{i=0}^{k} t_i(f(x_i) - f^*) \leq \frac{\|x_0 - x^*\|_2^2}{2} + \frac{1}{2}\sum_{i=0}^{k} t_i^2\|g_i\|_2^2.$$

Let $f_{\text{best},k} = \min\{f(x_0), \ldots, f(x_k)\}$. Since $t_i \geq 0$ we get

$$\begin{aligned} f_{\text{best},k} - f^* \leq \frac{1}{\sum t_i}\sum_{i=0}^{k} t_i(f(x_i) - f^*) &\leq \frac{\|x_0 - x^*\|_2^2}{2\sum_{i=0}^{k} t_i} + \frac{\sum_{i=0}^{k} t_i^2\|g_i\|_2^2}{2\sum_{i=0}^{k} t_i} \\ &\leq \frac{\|x_0 - x^*\|_2}{2\sum_{i=0}^{k} t_i} + \frac{G^2\sum_{i=0}^{k} t_i^2}{2\sum_{i=0}^{k} t_i}. \end{aligned} \tag{3}$$

where in the last equation we assumed that f is G-Lipschitz, so that $\|g_i\|_2 \leq G$ (see Exercise sheet 2).

- Constant step size: If $t_k = t$ and f is G-Lipschitz then we get

$$f_{\text{best},k} - f^* \leq \frac{\|x_0 - x^*\|_2^2}{2(k+1)t} + \frac{G^2 t}{2}. \tag{4}$$

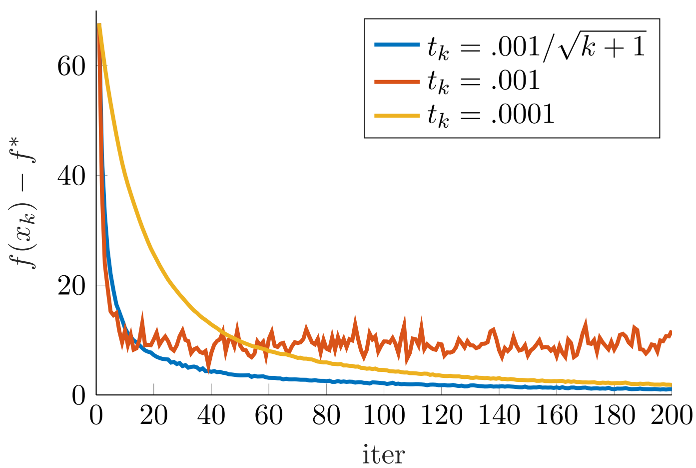
In this case we do not guarantee convergence: we only guarantee that $f_{\text{best},k}$ will be at most $G^2t/2$ sub-optimal, in the limit $k \to \infty$.

Assume that k is fixed a priori (i.e., we have a certain number of iterations that we are going to run). What is the choice of t that minimizes the right-hand side of (4)? The choice of t is the one that will make the two terms equal, namely $\|x_0 - x^*\|_2^2/(k+1) = G^2t^2$, i.e., $t = \|x_0 - x^*\|_2/(G\sqrt{k+1})$ and the corresponding bound we get with this choice of t is

$$t = \frac{\|x_0 - x^*\|_2}{G\sqrt{k+1}} \quad \Rightarrow \quad f_{\text{best},k} - f^* \leq \frac{G\|x_0 - x^*\|_2}{\sqrt{k+1}}.$$

- Diminishing step size: consider the choice $t_i \sim 1/\sqrt{i}$. Then $\sum_0^k t_i \approx \sqrt{k}$, $\sum_0^k t_i^2 \approx \ln(k)$, and so we get a convergence like $\ln(k)/\sqrt{k}$. In fact, one can get rid of the log term by recursing the inequality (1) only up to iterate $k/2$ (instead of all the way back to the first iterate), and use the fact that $\sum_{k/2}^k 1/i \leq$ constant.

Illustration The figure below shows the subgradient method applied to the problem of minimizing the nonsmooth function $f(x) = \|Ax - b\|_1$ where $A \in \mathbb{R}^{m \times n}$ with $m > n$, and $b \in \mathbb{R}^m$. We see that with a constant step size, the method does not converge to f^*, but only to a neighborhood of the optimal value.



Optimality of subgradient method One can show that the convergence rate of $1/\sqrt{k}$ is the best possible one can get on the class of nonsmooth convex Lipschitz functions. More precisely, fix k, G, and $R > 0$. For any algorithm where the k'th iterate satisfies

$$x_k \in x_0 + \text{span}\{g_1, \ldots, g_k\}$$

where $g_i \in \partial f(x_i)$ and x_0 is the starting point, there is a convex function f that is G-Lipschitz on $\{x : \|x - x_0\|_2 \leq R\}$ such that after k iterations of the algorithm we have

$$f_{\text{best},k} - f^* \gtrapprox \frac{GR}{\sqrt{k+1}}.$$

See Exercise sheet 2 for a proof.