4 Semidefinite programming

The vector space $\mathbf{S}^n$ of $n \times n$ real symmetric matrices is endowed with the *trace inner product*:

$$\langle A, B \rangle := \mathrm{Tr}(AB) = \sum_{1 \le i,j \le n} A_{ij} B_{ij}.$$

Recall the notation:

$$A \succeq 0 \iff A \text{ positive semidefinite.}$$

We are also going to use the following convenient notation:

$$A \succeq B \iff A - B \succeq 0.$$

A *semidefinite program* (SDP) is an optimisation problem of the form

$$\begin{array}{ll} \underset{X \in \mathbf{S}^n}{\text{minimise}} & \langle C, X \rangle \\ \text{subject to} & \mathcal{A}(X) = b \\ & X \succeq 0. \end{array} \tag{1}$$

where $C \in \mathbf{S}^n$, $\mathcal{A} : \mathbf{S}^n \to \mathbb{R}^m$ is a linear map and $b \in \mathbb{R}^m$. The optimisation variable is $X \in \mathbf{S}^n$. The constraint $X \succeq 0$ means that X is positive semidefinite. A semidefinite program (1) is entirely specified by the data $\mathcal{A} : \mathbf{S}^n \to \mathbb{R}^m, b \in \mathbb{R}^m, C \in \mathbf{S}^n$.

Example 1. *Consider the problem*

$$\underset{X \in \mathbf{S}^3}{\textit{minimise}} \quad X_{12} + X_{13} \quad \textit{s.t.} \quad X \succeq 0, \;\; \mathrm{diag}(X) = (1,1,1). \tag{2}$$

This is a semidefinite program of the form (1) where

$$C = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{A} : X \mapsto (X_{11}, X_{22}, X_{33}), \quad b = (1,1,1).$$

Problems of type (1) are usually called in semidefinite programming *standard form.* A semidefinite program in *linear matrix inequality form* is an optimisation problem:

$$\begin{array}{ll} \underset{x \in \mathbb{R}^k}{\text{minimise}} & c^T x \\ \text{subject to} & A_0 + x_1 A_1 + \cdots + x_k A_k \succeq 0 \end{array} \tag{3}$$

where $A_0, \ldots, A_k \in \mathbf{S}^n$ are given matrices and $c \in \mathbb{R}^k$ is a cost vector. It is not difficult to see that any problem of the form (1) can be put in the form (3), and vice-versa. We leave it as an exercise to the reader.

Example 2. *Consider*

$$\underset{x_1, x_2 \in \mathbb{R}}{\textit{minimise}} \quad x_1 + 2x_2 \quad \textit{s.t.} \quad \begin{bmatrix} 1 - x_1 & x_2 \\ x_2 & 1 + x_1 \end{bmatrix} \succeq 0. \tag{4}$$

This problem is readily put in the form (3) where $c = (1,2) \in \mathbb{R}^2$, $A_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), A_1 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The feasible set of a semidefinite program is a convex set of the form:

$$C = \{x \in \mathbb{R}^k : A_0 + x_1 A_1 + \cdots + x_k A_k \succeq 0\} \tag{5}$$

where $A_0, \ldots, A_k$ are $n \times n$ real symmetric matrices. Such a convex set is called a *spectrahedron*. If the matrices $A_0, \ldots, A_k$ are diagonal then C is a polyhedron; however spectrahedra are not polyhedral in general. For example the unit disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ (which is not polyhedral) is a spectrahedron. Indeed we have:

$$D = \left\{(x, y) \in \mathbb{R}^2 : \begin{bmatrix} 1-x & y \\ y & 1+x \end{bmatrix} \succeq 0 \right\}.$$

Deciding which convex sets are spectrahedra (i.e., admit a *linear matrix inequality* representation of the form (5)) is an open research question and there is currently no known simple necessary and sufficient conditions (some nontrivial necessary conditions are known however). For the curious reader we refer to Chapter 6 of [BPT12] for more details on this.

Example: ℓ_4 norm Consider the problem of optimising a linear function on the ℓ_4 norm ball:

$$\min_{x,y} ax + by \quad \text{s.t.} \quad x^4 + y^4 \leq 1. \tag{6}$$

This problem can be put in semidefinite programming form. Indeed note the following is true:

$$\begin{aligned} x^4 + y^4 \leq 1 &\iff \exists u, v \text{ s.t. } x^2 \leq u, y^2 \leq v, u^2 + v^2 \leq 1 \\ &\iff \exists u, v \text{ s.t. } \begin{bmatrix} u & x \\ x & 1 \end{bmatrix} \succeq 0, \begin{bmatrix} v & y \\ y & 1 \end{bmatrix} \succeq 0 \\ &\qquad\qquad\qquad \begin{bmatrix} 1-u & v \\ v & 1+u \end{bmatrix} \succeq 0. \end{aligned}$$

Thus problem (6) can be equivalently rewritten as the following SDP:

$$\min_{x,y,u,v} ax + by \quad \text{s.t.} \quad \begin{bmatrix} u & x \\ x & 1 \end{bmatrix} \succeq 0, \begin{bmatrix} v & y \\ y & 1 \end{bmatrix} \succeq 0, \begin{bmatrix} 1-u & v \\ v & 1+u \end{bmatrix} \succeq 0.$$

Solving semidefinite programs What makes the class of *semidefinite optimisation problems* interesting is that there are efficient computer algorithms (e.g., *interior-point methods*) to solve them, given the input data $\mathcal{A}, b, c$ for (1). We will not discuss these algorithms in this course. These algorithms have been implemented and interfaced with user-friendly modelling languages. In practice one can solve instances of (1) for n up to ≈ 100 in a reasonable amount of time on a personal computer. Larger instances can be solved by exploiting structure or using specialized algorithms. We recommend the MATLAB packages CVX (`http://www.cvxr.com`) or YALMIP (`https://yalmip.github.io/`) as a starting point. For example the semidefinite program (4) can be solved using the following code on MATLAB, using the modelling language CVX:

```
cvx_begin sdp
   variables x1 x2
   minimize x1 + 2*x2
   subject to
    [1-x1   x2;
       x2  1+x1] >= 0;
cvx_end
```

Example: (symmetric) matrix completion Consider the following (symmetric) *matrix completion*: we observe certain entries of an unknown symmetric matrix and the goal is to recover the symmetric matrix with the smallest *nuclear norm*. The nuclear norm of a symmetric matrix X is defined as the sum of the absolute values of the eigenvalues:

$$\|X\|_{\text{nuc}} = \sum_{i=1}^{n} |\lambda_i(X)|$$

where $\lambda_1(X), \ldots, \lambda_n(X)$ are the eigenvalues of X. The *nuclear norm* is also sometimes called the *trace norm* of X, or the *Schatten 1-norm*. It can be interpreted as the ℓ_1 norm of the eigenvalues of X. Let $\Omega \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ be the subset of entries that we observe and let M_{ij} be the values that we observe. We want to solve the problem:

$$\underset{X \in \mathbf{S}^n}{\text{minimise}} \quad \|X\|_{\text{nuc}} \quad \text{s.t.} \quad X_{ij} = M_{ij} \ \forall (i,j) \in \Omega. \tag{7}$$

We now show how to formulate (7) as a semidefinite program. Consider the following semidefinite program:

$$\underset{X, Y \in \mathbf{S}^n}{\text{minimise}} \quad \text{Tr}(Y) \quad \text{s.t.} \quad X_{ij} = M_{ij} \ \forall (i,j) \in \Omega, \quad Y - X \succeq 0, \quad Y + X \succeq 0. \tag{8}$$

The next claim shows that the two problems (7) and (8) are "equivalent".

Claim 4.1. *Assume X is feasible for (7). Then there exists $Y \in \mathbf{S}^n$ such that (X, Y) is feasible for (8) and $\text{Tr}(Y) \le \|X\|_{\text{nuc}}$. Conversely if (X, Y) is feasible for (8) then $\|X\|_{\text{nuc}} \le \text{Tr}(Y)$. As a consequence, the optimal values of (7) and (8) are equal.*

Proof. For the first direction, assuming $X = \sum_{i=1}^n \lambda_i v_i v_i^T$ is an eigendecomposition of X, we let $Y = \sum_{i=1}^n |\lambda_i| v_i v_i^T$. Note that $\text{Tr}(Y) = \|X\|_{\text{nuc}}$. We need to show that $Y - X \succeq 0$ and $Y + X \succeq 0$. Note that $Y \pm X = \sum_{i=1}^n (|\lambda_i| \pm \lambda_i) v_i v_i^T$ and thus is positive semidefinite since $|\lambda_i| \pm \lambda_i \ge 0$.

For the converse, assume (X, Y) satisfy the constraints of (8). We need to show that $\|X\|_{\text{nuc}} \le \text{Tr}(Y)$. Let $X = \sum_{i=1}^n \lambda_i v_i v_i^T$ be an eigenvalue decomposition of X. Let $P^+ = \sum_{i:\lambda_i \ge 0} v_i v_i^T$ and $P^- = \sum_{i:\lambda_i < 0} v_i v_i^T$. Note that $P^+ + P^- = I_n$ and $\text{Tr}(X(P^+ - P^-)) = \|X\|_{\text{nuc}}$. Since $Y - X \succeq 0$ and $P^+ \succeq 0$ we have $\text{Tr}((Y - X)P^+) \ge 0$. Similarly we have $\text{Tr}((Y + X)P^-) \ge 0$. Adding these two inequalities we get $\text{Tr}(Y) - \|X\|_{\text{nuc}} \ge 0$ which is what we wanted. □

References

[BPT12] Grigoriy Blekherman, Pablo A. Parrilo, and Rekha R. Thomas. *Semidefinite optimization and convex algebraic geometry*. SIAM, 2012. 2