Mathematical Tripos Part II: Michaelmas Term 2021

Numerical Analysis – Lecture 5

2 Partial differential equations of evolution

Definition 2.1 (Stability in the context of time-stepping methods for PDEs of evolution) A numerical method for a PDE of evolution is *stable* if (for zero boundary conditions) it produces a uniformly bounded approximation of the solution in any bounded interval of the form $0 \le t \le T$ when $h \to 0$ and the generalized Courant number $\mu = k/h^r$, with r being the maximum degree of the differential operator, is constant.

This definition is relevant not just for the diffusion equation but for every PDE of evolution which is *well-posed*, i.e. such that its exact solution depends (in a compact time interval) in a uniformly bounded manner on the initial conditions. Thus, "stability" is nothing but the statement that well-posedness is retained under discretization, uniformly for $h \to 0$. Most PDEs of practical interest are well-posed.

Theorem 2.2 (The Lax equivalence theorem) *Suppose that the underlying PDE is well-posed and that it is solved by a numerical method with an error of $\mathcal{O}(h^{p+r})$, $p \ge 1$, where r is the maximum degree of the differential operator. Then stability $\Leftrightarrow$ convergence.*

Method 2.3 (Analysis of stability) Suppose that a numerical method (with zero boundary conditions) can be written in the form

$$\boldsymbol{u}_h^{n+1} = A_h \boldsymbol{u}_h^n,$$

where $\boldsymbol{u}_h^n \in \mathbb{R}^M$ are vectors, $A_h \in \mathbb{R}^{M\times M}$ is a matrix, and $h = \frac{1}{M+1}$. Then $\boldsymbol{u}_h^n = (A_h)^n \boldsymbol{u}_h^0$, and

$$\|\boldsymbol{u}_h^n\| = \|(A_h)^n \boldsymbol{u}_h^0\| \le \|(A_h)^n\| \cdot \|\boldsymbol{u}_h^0\| \le \|A_h\|^n \cdot \|\boldsymbol{u}_h^0\|,$$

for any vector norm $\|\cdot\|$ and the induced matrix norm $\|A\| = \sup \frac{\|A\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$. If we define stability as preserving the boundedness of $\boldsymbol{u}_h^n$ with respect to the norm $\|\cdot\|$, then, from the inequality above,

$$\|A_h\| \le 1 \text{ as } h \to 0 \quad \Rightarrow \quad \text{the method is stable.}$$

In the proof of Theorem 2.2, we used the infinity norm

$$\|\boldsymbol{u}\|_\infty = \max_{i=1,\dots,M} |u_i|.$$

It can be easily shown that the corresponding induced norm for a matrix $A \in \mathbb{R}^{M\times M}$ is given by:

$$\|A\|_{\infty\to\infty} := \sup_{\boldsymbol{x}} \frac{\|A\boldsymbol{x}\|_\infty}{\|\boldsymbol{x}\|_\infty} = \max_{i=1,\dots,M} \sum_{j=1}^M |A_{ij}|.$$

Another common of choice of norm is the averaged Euclidean length, namely, $\|\boldsymbol{u}\|_h := [h \sum_{i=1}^M |u_i|^2]^{1/2}$. The reason for the factor $h^{1/2}$ is to ensure that, because of the convergence of Riemann sums, we obtain

$$\|\boldsymbol{u}\|_h := \Big[h \textstyle\sum_{i=1}^M |u_i|^2\Big]^{1/2} \to \Big[\int_0^1 |u(x)|^2 \mathrm{d}x\Big]^{1/2} =: \|u\|_{L_2} \qquad (h \to 0),$$

The induced matrix norm in this case is the *spectral norm* (or the *operator norm*) and is denoted $\|A\|_2$:[1]

$$\|A\|_2 := \sup_{\boldsymbol{x}} \frac{\|A\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2}.$$

[1] Note that $\|A\boldsymbol{x}\|_h / \|\boldsymbol{x}\|_h = \|A\boldsymbol{x}\|_2 / \|\boldsymbol{x}\|_2$ where $\|x\|_2 = (\sum_i |x_i|^2)^{1/2}$ is the usual Euclidean norm

The spectral norm of A is equal to the largest singular value of A. Equivalently, we can write $\|A\|_2 = [\rho(A\bar{A}^T)]^{1/2}$ where ρ is the spectral radius:

$$\rho(M) := \max\{|\lambda| : \lambda \text{ eigenvalue of } M\}.$$

For certain matrices, such as normal matrices, one can show that $\|A\|_2 = \rho(A)$.

Definition 2.4 (Normal matrices) A complex matrix $A \in \mathbb{C}^{M\times M}$ is *normal* if it commutes with its conjugate transpose, i.e., $A\bar{A}^T = \bar{A}^T A$.

Examples of real normal matrices include symmetric matrices ($A = A^T$) and skew-symmetric matrices ($A = -A^T$). Any normal matrix A can be diagonalized in an orthonormal basis, i.e., $A = QDQ^T$ where Q unitary, $Q\bar{Q}^T = \bar{Q}^T Q = I$, and D is diagonal. Note however that the diagonal elements D_{ii} are not necessarily real!

Proposition 2.5 *If A is normal, then $\|A\|_2 = \rho(A)$.*

Proof. Let $\boldsymbol{u}$ be any vector. We can expand it in the basis of the orthonormal eigenvectors $\boldsymbol{u} = \sum_{i=1}^n a_i \boldsymbol{q}_i$. Then $A\boldsymbol{u} = \sum_{i=1}^n \lambda_i a_i \boldsymbol{q}_i$, and since $\boldsymbol{q}_i$ are orthonormal, we obtain

$$\|A\|_2 := \sup_{\boldsymbol{u}} \frac{\|A\boldsymbol{u}\|_2}{\|\boldsymbol{u}\|_2} = \sup_{a_i} \frac{\{\sum_{i=1}^n |\lambda_i a_i|^2\}^{1/2}}{\{\sum_{i=1}^n |a_i|^2\}^{1/2}} = |\lambda_{\max}|.$$

Example 2.6 (Stability of (2.2)) We can analyze the stability of (2.2) using the eigenvalue methods just described. The recurrence (2.2) can be written as:

$$u_m^{n+1} = u_m^n + \mu\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right), \qquad m = 1...M,$$

in the matrix form

$$\boldsymbol{u}_h^{n+1} = A_h \boldsymbol{u}_h^n, \qquad A_h = I + \mu A_*, \qquad A_* = \begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}_{M\times M}.$$

Here A_* is Toeplitz, symmetric, tridiagonal (TST), with $\lambda_\ell(A_*) = -4\sin^2\frac{\pi\ell h}{2}$, hence $\lambda_\ell(A_h) = 1 - 4\mu\sin^2\frac{\pi\ell h}{2}$, so that its spectrum lies within the interval $[\lambda_M, \lambda_1] = [1 - 4\mu\cos^2\frac{\pi h}{2}, 1 - 4\mu\sin^2\frac{\pi h}{2}]$. Since A_h is symmetric, we have

$$\|A_h\|_2 = \rho(A_h) = \begin{cases} |1 - 4\mu\sin^2\frac{\pi h}{2}| \le 1, & \mu \le \frac{1}{2}, \\ |1 - 4\mu\cos^2\frac{\pi h}{2}| > 1, & \mu > \frac{1}{2} \quad (h \le h_\mu). \end{cases}$$

We distinguish between two cases.

1) $\mu \le \frac{1}{2}$: $\|\boldsymbol{u}^n\| \le \|A\| \cdot \|\boldsymbol{u}^{n-1}\| \le \cdots \le \|A\|^n \|\boldsymbol{u}^0\| \le \|\boldsymbol{u}^0\|$ as $n \to \infty$, for every $\boldsymbol{u}^0$.

2) $\mu > \frac{1}{2}$: Choose $\boldsymbol{u}^0$ as the eigenvector corresponding to the largest (in modulus) eigenvalue, $|\lambda| > 1$. Then $\boldsymbol{u}^n = \lambda^n \boldsymbol{u}^0$, becoming unbounded as $n \to \infty$.