

Mathematical Tripos Part II: Michaelmas Term 2021

Numerical Analysis – Lecture 7

Technique 2.11 (Fourier analysis of stability) Let us now assume a recurrence of the form

$$\sum_{k=r}^s a_k u_{m+k}^{n+1} = \sum_{k=r}^s b_k u_{m+k}^n, \quad n \in \mathbb{Z}^+, \quad (2.5)$$

where m ranges over \mathbb{Z} . (Within our framework of discretizing PDEs of evolution, this corresponds to $-\infty < x < \infty$ in the underlying PDE and so there are no explicit boundary conditions, but the initial condition must be square-integrable in $(-\infty, \infty)$: this is known as a *Cauchy problem*.) The coefficients a_k and b_k are independent of m, n , but typically depend upon μ . We investigate stability by *Fourier analysis*. [Note that it doesn't matter what is the underlying PDE: numerical stability is a feature of algebraic recurrences, not of PDEs!]

Let $\mathbf{v} = (v_m)_{m \in \mathbb{Z}} \in \ell_2[\mathbb{Z}]$. Its *Fourier transform* is the function

$$\widehat{v}(\theta) = \sum_{m \in \mathbb{Z}} e^{-im\theta} v_m, \quad -\pi \leq \theta \leq \pi.$$

We equip sequences and functions with the norms

$$\|\mathbf{v}\| = \left\{ \sum_{m \in \mathbb{Z}} |v_m|^2 \right\}^{\frac{1}{2}} \quad \text{and} \quad \|\widehat{v}\|_* = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widehat{v}(\theta)|^2 d\theta \right\}^{\frac{1}{2}}.$$

Lemma 2.12 (Parseval's identity) For any $\mathbf{v} \in \ell_2[\mathbb{Z}]$, we have $\|\mathbf{v}\| = \|\widehat{v}\|_*$.

Proof. By definition,

$$\begin{aligned} \|\widehat{v}\|_*^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{m \in \mathbb{Z}} e^{-im\theta} v_m \right|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} v_m \bar{v}_k e^{-i(m-k)\theta} d\theta \\ &= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} v_m \bar{v}_k \int_{-\pi}^{\pi} e^{-i(m-k)\theta} d\theta \stackrel{(*)}{=} \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} v_m \bar{v}_k \delta_{m-k} = \|\mathbf{v}\|^2, \end{aligned}$$

where equality $(*)$ is due to the fact that

$$\int_{-\pi}^{\pi} e^{-i\ell\theta} d\theta = \begin{cases} 2\pi, & \ell = 0, \\ 0, & \ell \in \mathbb{Z} \setminus \{0\}, \end{cases} \quad \square$$

The implication of the lemma is that the Fourier transform is an *isometry* of the Euclidean norm. This is an important reason underlying its many applications in mathematics and beyond.

Analysis 2.13 (Fourier analysis of stability) For $\theta \in [-\pi, \pi]$, let $\widehat{u}^n(\theta) = \sum_{m \in \mathbb{Z}} e^{-im\theta} u_m^n$ be the Fourier transform of the sequence $\mathbf{u}^n \in \ell_2[\mathbb{Z}]$. We multiply the discretized equations (2.5) by $e^{-im\theta}$ and sum up for $m \in \mathbb{Z}$. Thus, the left-hand side yields

$$\begin{aligned} \sum_{m=-\infty}^{\infty} e^{-im\theta} \sum_{k=r}^s a_k u_{m+k}^{n+1} &= \sum_{k=r}^s a_k \sum_{m=-\infty}^{\infty} e^{-im\theta} u_{m+k}^{n+1} \\ &= \sum_{k=r}^s a_k \sum_{m=-\infty}^{\infty} e^{-i(m-k)\theta} u_m^{n+1} = \left(\sum_{k=r}^s a_k e^{ik\theta} \right) \widehat{u}^{n+1}(\theta). \end{aligned}$$

Similarly manipulating the right-hand side, we deduce that

$$\widehat{u}^{n+1}(\theta) = H(\theta) \widehat{u}^n(\theta), \quad \text{where} \quad H(\theta) = \frac{\sum_{k=r}^s b_k e^{ik\theta}}{\sum_{k=r}^s a_k e^{ik\theta}}. \quad (2.6)$$

The function H is sometimes called the *amplification factor* of the recurrence (2.5)

Theorem 2.14 The method (2.5) is stable $\Leftrightarrow |H(\theta)| \leq 1$ for all $\theta \in [-\pi, \pi]$.

Proof. The definition of stability is equivalent to the statement that there exists $c > 0$ such that $\|\mathbf{u}^n\| \leq c$ for all $n \in \mathbb{Z}^+$. [Because we are solving a Cauchy problem, equations are identical for all $h = \Delta x$, and this simplifies our analysis and eliminates a major difficulty: there is no need to insist explicitly that $\|\mathbf{u}^n\|$ remains uniformly bounded when $h \rightarrow 0$]. The Fourier transform being an isometry, stability is thus equivalent to $\|\hat{u}^n\|_* \leq c$ for all $n \in \mathbb{Z}^+$. Iterating (2.6), we obtain

$$\hat{u}^n(\theta) = [H(\theta)]^n \hat{u}^0(\theta), \quad |\theta| \leq \pi, \quad n \in \mathbb{Z}^+. \quad (2.7)$$

1) Assume first that $|H(\theta)| \leq 1$ for all $|\theta| \leq \pi$. Then, by (2.7),

$$|\hat{u}^n(\theta)| \leq |\hat{u}^0(\theta)| \quad \Rightarrow \quad \|\hat{u}^n\|_*^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{u}^n(\theta)|^2 d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{u}^0(\theta)|^2 d\theta = \|\hat{u}^0\|_*^2.$$

Hence stability.

2) Suppose, on the other hand, that there exists $\theta_0 \in [-\pi, \pi]$ such that $|H(\theta_0)| = 1 + 2\epsilon > 1$, say. Since H is continuous, there exist $-\pi \leq \theta_1 < \theta_2 \leq \pi$ such that $|H(\theta)| \geq 1 + \epsilon$ for all $\theta \in [\theta_1, \theta_2]$. We set $\eta = \theta_2 - \theta_1$ and choose as our initial condition the function (or the $\ell_2[\mathbb{Z}]$ -sequence)

$$\hat{u}^0(\theta) = \begin{cases} \sqrt{\frac{2\pi}{\eta}}, & \theta_1 \leq \theta \leq \theta_2, \\ 0, & \text{otherwise,} \end{cases}$$

Then

$$\begin{aligned} \|\hat{u}^n\|_*^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\theta)|^{2n} |\hat{u}^0(\theta)|^2 d\theta = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} |H(\theta)|^{2n} |\hat{u}^0(\theta)|^2 d\theta \\ &\geq \frac{1}{2\pi} (1 + \epsilon)^{2n} \int_{\theta_1}^{\theta_2} \frac{2\pi}{\eta} d\theta = (1 + \epsilon)^{2n} \rightarrow \infty \quad (n \rightarrow \infty). \end{aligned}$$

We deduce that the method is unstable. □

Example 2.15 Consider the Cauchy problem for the diffusion equation.

1) For the Euler method

$$u_m^{n+1} = u_m^n + \mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n),$$

we obtain

$$H(\theta) = 1 + \mu(e^{-i\theta} - 2 + e^{i\theta}) = 1 - 4\mu \sin^2 \frac{\theta}{2} \in [1 - 4\mu, 1],$$

thus the method is stable iff $\mu \leq \frac{1}{2}$.

2) For the backward Euler method

$$u_m^{n+1} - \mu(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n,$$

we have

$$H(\theta) = [1 - \mu(e^{-i\theta} - 2 + e^{i\theta})]^{-1} = [1 + 4\mu \sin^2 \frac{\theta}{2}]^{-1} \in (0, 1].$$

thus stability for all μ .

3) The Crank–Nicolson scheme

$$u_m^{n+1} - \frac{1}{2}\mu(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \frac{1}{2}\mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n),$$

results in

$$H(\theta) = \frac{1 + \frac{1}{2}\mu(e^{-i\theta} - 2 + e^{i\theta})}{1 - \frac{1}{2}\mu(e^{-i\theta} - 2 + e^{i\theta})} = \frac{1 - 2\mu \sin^2 \frac{\theta}{2}}{1 + 2\mu \sin^2 \frac{\theta}{2}} \in (-1, 1]$$

Hence stability for all $\mu > 0$.