3 The positive semidefinite cone

In this course we will focus a lot of our attention on the *positive semidefinite cone*. Let $\mathbf{S}^n$ denote the vector space of $n \times n$ real symmetric matrices. Recall that by the spectral theorem any matrix $A \in \mathbf{S}^n$ is diagonalisable in an orthonormal basis and has real eigenvalues. Let $\mathbf{S}^n_+$ (resp. $\mathbf{S}^n_{++}$) denote the set of positive semidefinite matrices, i.e., the set of real symmetric matrices having nonnegative (resp. strictly positive) eigenvalues. For a matrix $A \in \mathbf{S}^n_+$ we will use the following convenient notations:

$$A \succeq 0 \iff A \text{ positive semidefinite}$$

and

$$A \succ 0 \iff A \text{ positive definite.}$$

Proposition 3.1. *Let $A \in \mathbf{S}^n$. The following conditions are equivalent:*

(i) $A \in \mathbf{S}^n_+$

(ii) The eigenvalues of A are nonnegative

(iii) $x^T A x \geq 0$ *for all* $x \in \mathbb{R}^n$

(iv) There exists $L \in \mathbb{R}^{n\times n}$ lower triangular such that $A = LL^T$ (Cholesky factorization)

(v) All the principal minors of A are nonnegative, i.e., $\det A[S,S] \geq 0$ *for any nonempty $S \subseteq \{1,\ldots,n\}$ where $A[S,S]$ is the submatrix of A consisting of the rows and columns indexed by S (Sylvester criterion)*

Also the following are all equivalent:

(i) $A \in \mathbf{S}^n_{++}$

(ii) The eigenvalues of A are strictly positive

(iii) $x^T A x > 0$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$

(iv) There exists $L \in \mathbb{R}^{n\times n}$ lower triangular with $L_{ii} > 0$ for all $i = 1,\ldots,n$ such that $A = LL^T$ (Cholesky factorization)

(v) All the leading principal minors of A are positive, i.e., $\det A[\{1,\ldots,k\},\{1,\ldots,k\}] > 0$ *for all $k = 1,\ldots,n$ (Sylvester criterion)*

Proof. We leave it as an exercise to the reader. □

Theorem 3.1. *$\mathbf{S}^n_+$ is a closed pointed convex cone in $\mathbf{S}^n$ with interior $\mathbf{S}^n_{++}$.*

Proof. That $\mathbf{S}^n_+$ is closed and convex follows from

$$\mathbf{S}^n_+ = \{A \in \mathbf{S}^n : x^T A x \geq 0 \; \forall x \in \mathbb{R}^n\} = \bigcap_{x \in \mathbb{R}^n} \underbrace{\{A \in \mathbf{S}^n : x^T A x \geq 0\}}_{H_x}.$$

H_x is a closed halfspace in $\mathbf{S}^n$ for any fixed x thus $\mathbf{S}^n_+$ is closed and convex as an intersection of closed halfspaces. To show that $\mathbf{S}^n_+$ is pointed we need to show that $(\mathbf{S}^n_+) \cap (-\mathbf{S}^n_+) = \{0\}$. This is easy to see because if $A \in \mathbf{S}^n_+ \cap (-\mathbf{S}^n_+)$ then all the eigenvalues of A must be equal to zero which means that $A = 0$. It remains to show that $\text{int}(\mathbf{S}^n_+) = \mathbf{S}^n_{++}$. To do so we first define the *spectral norm* of a matrix $A \in \mathbf{S}^n$ as:

$$\|A\| = \max_{x \in \mathbb{R}^n : \|x\|_2 = 1} \|Ax\|_2 = \max\{-\lambda_{\min}(A), \lambda_{\max}(A)\}.$$

Note that this is the $\ell_2 \to \ell_2$ induced norm. We now show that $\text{int}(\mathbf{S}^n_+) = \mathbf{S}^n_{++}$.

- We first show the inclusion $\text{int}(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_{++}$. If $A \in \text{int}(\mathbf{S}^n_+)$ then there exists small enough $\epsilon > 0$ such that $\|A - X\| \leq \epsilon \Rightarrow X \in \mathbf{S}^n_+$. Let $X = A - \epsilon I$ where I is the $n \times n$ identity matrix, and note that $\|A - X\| = \|\epsilon I\| \leq \epsilon$. It thus follows that $X = A - \epsilon I \in \mathbf{S}^n_+$. Since the eigenvalues of $A - \epsilon I$ are the $(\lambda_i - \epsilon)$ (where (λ_i) are the eigenvalues of A) it follows that $\lambda_i \geq \epsilon > 0$ and thus A is positive definite, i.e., $A \in \mathbf{S}^n_{++}$.
- We now prove the reverse inclusion $\mathbf{S}^n_{++} \subseteq \text{int}(\mathbf{S}^n_+)$. Let $A \in \mathbf{S}^n_{++}$. Let $\lambda_{\min} > 0$ be the smallest eigenvalue of A and define the spectral norm ball $B = \{M \in \mathbf{S}^n : \|M - A\| \leq \lambda_{\min}/2\}$. We will show that the ball B is included in $\mathbf{S}^n_+$ which will establish our claim. Let M such that $\|M - A\| \leq \lambda_{\min}/2$. Then this means that for any x with $\|x\| = 1$, $x^T(A - M)x \leq \lambda_{\min}/2$ and so $x^T M x \geq x^T A x - \lambda_{\min}/2 \geq \lambda_{\min}/2 > 0$. We have shown that $x^T M x \geq 0$ for any unit normed x thus M is positive semidefinite. This completes the proof.

□

The real vector space $\mathbf{S}^n$ has dimension $\binom{n+1}{2}$. We equip this vector space with the (trace) inner product

$$\langle A, B \rangle := \text{Tr}[AB] = \sum_{1 \leq i,j \leq n} A_{ij} B_{ij}.$$

With this inner product the cone $\mathbf{S}^n_+$ is self-dual, meaning that $(\mathbf{S}^n_+)^* = \mathbf{S}^n_+$.

Theorem 3.2. *With the trace inner product on* $\mathbf{S}^n$ *we have* $(\mathbf{S}^n_+)^* = \mathbf{S}^n_+$.

Proof. By definition $(\mathbf{S}^n_+)^* = \{B \in \mathbf{S}^n : \text{Tr}(AB) \geq 0 \ \forall A \in \mathbf{S}^n_+\}$. We first show that $\mathbf{S}^n_+ \subseteq (\mathbf{S}^n_+)^*$. Assume B is positive semidefinite. The eigenvalue decomposition of B takes the form $B = \sum_{i=1}^n \lambda_i v_i v_i^T$ where $\lambda_i \geq 0$ for $i = 1, \ldots, n$ and the v_i are the unit-normed eigenvectors of B. Now for any $A \in \mathbf{S}^n_+$ we have $\text{Tr}(AB) = \sum_{i=1}^n \lambda_i \text{Tr}(A v_i v_i^T) = \sum_{i=1}^n \lambda_i v_i^T A v_i$. Since $A \in \mathbf{S}^n_+$ we have $v_i^T A v_i \geq 0$ for all $i = 1, \ldots, n$ and thus, since $\lambda_i \geq 0$ we get $\text{Tr}(AB) \geq 0$. This shows $\mathbf{S}^n_+ \subseteq (\mathbf{S}^n_+)^*$.

To show the reverse inclusion, assume $B \in \mathbf{S}^n$ is such that $\text{Tr}(AB) \geq 0$ for all $A \in \mathbf{S}^n_+$. We want to show that B is positive semidefinite. By taking $A = xx^T$ for any $x \in \mathbb{R}^n$ we get that $\text{Tr}(xx^T B) = x^T B x \geq 0$. This is true for all $x \in \mathbb{R}^n$ and thus shows that B is positive semidefinite. □

Theorem 3.3. *The extreme rays of* $\mathbf{S}^n_+$ *are the rays spanned by rank-one matrices, i.e., of the form* $S_x = \{\lambda xx^T, \lambda \geq 0\}$ *where* $x \in \mathbb{R}^n$.

Proof. We first show that any ray spanned by a matrix of the form xx^T is extreme for $\mathbf{S}^n_+$. Then we will show that these are the only ones.

- Assume $A, B \in \mathbf{S}^n_+$ are such that $A + B = \lambda xx^T$ for some $\lambda \geq 0$. We need to show that A and B are both a multiple of xx^T. Let u be any vector orthogonal to x, i.e., $u^T x = 0$. Then $0 \leq u^T Au \leq u^T (A + B)u = u^T (\lambda xx^T)u = 0$. Thus for any $u \in \{x\}^\perp$ we have $u^T Au = 0$ which implies, since $A \succeq 0$, $u \in \ker(A)$ (see Exercise 3.2). Since $\operatorname{im}(A) = \ker(A)^\perp$ for any symmetric matrix A we get $\operatorname{im}(A) = \ker(A)^\perp \subseteq \operatorname{span}(x)$. This means that A is of the form $A = \lambda xx^T$. One can show in a similar way that B is a nonnegative multiple of xx^T.
- We now show that these are the only extreme rays. Consider a ray $S = \{\lambda A : \lambda \geq 0\}$ spanned by some matrix $A \in \mathbf{S}^n_+$. If $\operatorname{rank}(A) \geq 2$, an eigenvalue decomposition allows us to express A as a sum of elements that are not in S which shows that S cannot be an extreme ray.

□

Exercise 3.1. *Faces of the positive semidefinite cone.*

1. *Let V be a subspace of $\mathbb{R}^n$. Show that*

$$F_V = \left\{ Y \in \mathbf{S}^n_+ : \operatorname{im} Y \subseteq V \right\}$$

   is a face of the positive semidefinite cone. What is its dimension?

2. *Let $X \in \mathbf{S}^n_+$. Show that the smallest closed face of $\mathbf{S}^n_+$ containing X is $F_{\operatorname{im} X}$.*

Exercise 3.2. *Some properties of positive semidefinite matrices.*

- *Let $A \in \mathbf{S}^n_+$ and $u \in \mathbb{R}^n$. Show that $u^T Au = 0 \iff u \in \ker(A)$.*
- *Let $A \in \mathbf{S}^n$ and R an invertible $n \times n$ matrix. Show that $A \succeq 0 \iff R^T AR \succeq 0$ and $A \succ 0 \iff R^T AR \succ 0$.*
- *(Schur complement) Show that*

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \iff A \succ 0 \text{ and } C - B^T A^{-1} B \succ 0$$

- *Recall that we use the notation $A \succ B$ for $A - B \succ 0$. Show that if $A \succ B \succ 0$ then $A^{-1} \prec B^{-1}$ (Hint: start with the case $B = I$ (identity matrix) then use the fact that $A \succ B$ if and only $B^{-1/2} AB^{-1/2} \succ I$).*
- *(Schur product theorem) Let $A, B \in \mathbf{S}^n$ and assume that $A \succeq 0$ and $B \succeq 0$. Show that $A \odot B \succeq 0$ where $A \odot B$ is the* entrywise product *of A and B, i.e., $(A \odot B)_{ij} = A_{ij} B_{ij}$ (Hint: start with the case where A has rank one).*