

A note on the mechanism of the instability at the interface between two shearing fluids

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In a recent paper Hooper & Boyd (1983) have shown that the unbounded stratified Couette flow of two viscous fluids of equal density and with no interfacial tension is always unstable. They found that the instability arises at the interface between the two fluids and occurs at short wavelengths where viscosity is more important than inertia. The purpose of this note is to provide a simple physical explanation of the mechanics of the instability.

Consider two semi-infinite fluids in $y \leq 0$ with respective viscosities μ_{\pm} in a shearing motion $\mathbf{u} = (\gamma_{\pm}y, 0, 0)$. By continuity of the shear stress at the interface $y = 0$, $\mu_{+}\gamma_{+} = \mu_{-}\gamma_{-}$. We shall take the upper fluid to be more viscous, i.e. $\mu_{+} > \mu_{-}$.

Now consider the disturbance caused by displacing the interface with a small-amplitude wave from $y = 0$ to $y = \eta(x) = \eta_0 \cos kx$ (see figure 1). At the disturbed interface, at both the peaks *A* and the troughs *B*, the undisturbed velocity of the more viscous upper fluid is slower than that of the lower fluid. Thus the boundary condition that velocity is continuous requires that the upper fluid must speed up on the disturbed interface while the lower fluid must slow down relative to the undisturbed shear of the respective fluids. The magnitude of the speeding up of the upper fluid depends on the contrast in viscosity between the two fluids. If the viscosities are nearly equal, then the horizontal velocity at the disturbed interface takes the average value of the nearly equal undisturbed shears. If on the other hand the upper fluid is much more viscous, then the horizontal velocity is low and most of the accommodation has to be made by the less viscous lower fluid.

The adjustments of the horizontal velocity on the disturbed interface can be interpreted as disturbance velocities, which at the peaks *A* are positive in the upper fluid and negative in the lower fluid, while at the troughs *B* they are of the opposite signs. These disturbance velocities are maximum on the interface and decay away into the interior of the fluids. Thus we can convert the disturbance velocities into disturbance vorticities which are positive (anticlockwise) in both the upper and lower fluids at the peaks *A* and negative at the troughs *B* (see figure 1). Because the viscous stresses match at the interface, the vorticity will be larger in the less viscous lower fluid.

At short wavelengths, viscosity dominates inertia, and so vorticity diffuses easily. The disturbance (subtracting off the undisturbed shear) will therefore take the form of cells penetrating a distance of roughly half a wavelength, and rotating anticlockwise above and below the peaks *A* and clockwise above and below the troughs. Thus well away from the interface (at a distance k^{-1} roughly equal to $\frac{1}{6}$ of the wavelength) the horizontal disturbance velocity changes sign from its value at the interface. This consequence of mass conservation and the diffusion of vorticity dominating is indicated in figure 1, although it is incidental to the instability mechanism.

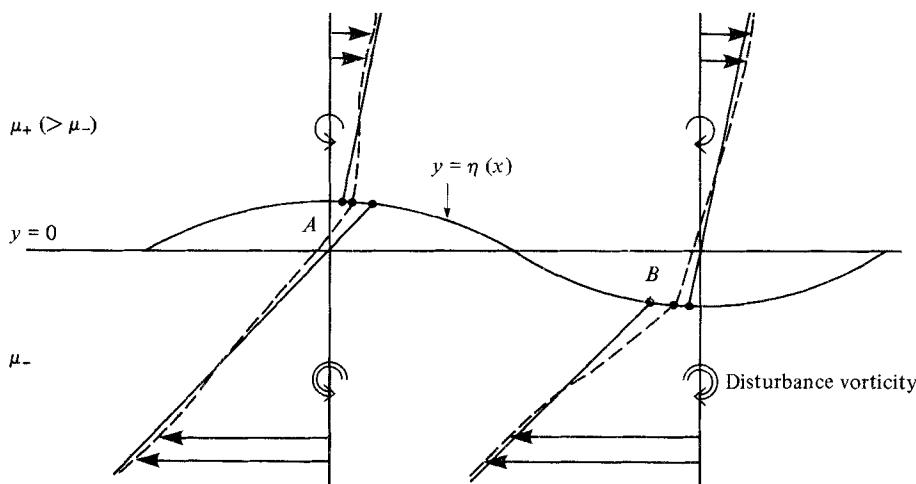


FIGURE 1. The horizontal velocity profiles. The continuous curves are the undisturbed shear, while the broken curves are the disturbed (i.e. undisturbed + disturbance) profiles.

We can now estimate the magnitude of the disturbance vorticity ω as the horizontal disturbance velocity divided by the vertical extent of the disturbance, i.e. k^{-1} as explained above. When the viscosities are greatly different then $\omega_{\pm} = O(\gamma_{\pm} \eta_0 k)$. On the other hand, if the viscosities are nearly equal then both ω_+ and ω_- are $O((\gamma_- - \gamma_+) \eta_0 k)$, and their difference is smaller again by a factor $(\mu_+ - \mu_-)(\mu_+ + \mu_-)$.

If inertia were entirely negligible, then the interface would not move (in a linear theory) and so there would be no instability. This observation follows from the reversibility of Stokes flows, which reverses any vertical motion of the interface when the direction of the shear flows is reversed. Thus the instability requires an examination of the effects of small inertia, $O(\rho\gamma/\mu k^2)$.

The small effects of inertia are best considered with use of the vorticity equation. The vorticity distribution derived above ignoring inertia will be advected by the basic shear to create small out-of-phase components midway between the peaks and troughs (see figure 2). With the advected vorticity being diffused away strongly, the magnitude of the out-of-phase components will be $\omega'_{\pm} = O(\rho\gamma_{\pm} \omega_{\pm} / \mu_{\pm} k^2)$. Thus the larger vorticity in the less viscous lower fluid produces a relatively larger out-of-phase component of vorticity.

We now consider the motion induced by the out-of-phase components of vorticity. As seen in figure 2, the out-of-phase component of vorticity in the lower fluid acting alone would induce an upward motion of the peaks and a downward motion of the troughs, while this vorticity in the upper fluid would induce the opposite motion. Because the out-of-phase component of vorticity is larger in the less viscous lower fluid, we conclude that the peaks will move up and the troughs down, i.e. the disturbance on the interface grows. Note that the instability does not propagate along the interface at this order of small inertial effects.

From the earlier estimates of the magnitude of the out-of-phase components of vorticity we can estimate the induced upward velocity of the peaks and hence the growth rate of the instability. When the viscosities are nearly equal, the upward velocity will be $O((\omega'_- - \omega'_+)/k)$, which yields a growth rate $O(\rho\gamma^2(\mu_+ - \mu_-)^2/k^2\mu^3)$. When the viscosity of the upper fluid is much larger than that of the lower fluid, the estimate $O(\omega'_-/k)$ for the vertical velocity in the lower fluid is too large at the interface. In order to move the interface, the very viscous upper fluid must also move. In response to the out-of-phase pressure disturbances $O(\mu_- \omega'_-)$ from the lower fluid, the

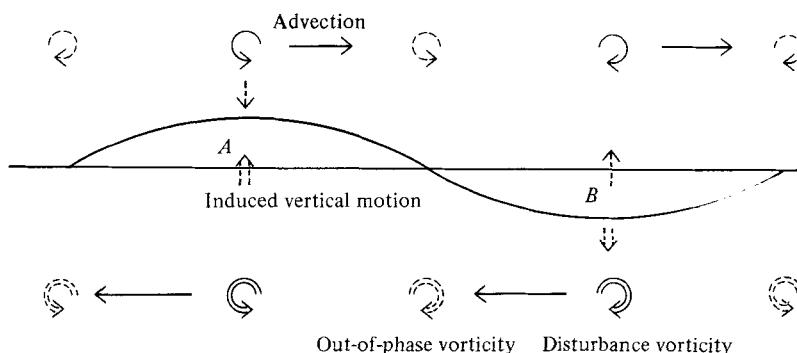


FIGURE 2. The disturbance vorticity, shown continuous, is advected by the mean shear to produce small out-of-phase components of vorticity, shown broken, which then induce a vertical motion of the interface.

upper fluid will respond with vertical velocities $O(\mu_- \omega'_- / \mu_+ k)$. This corresponds to a growth rate $O(\rho \gamma_-^2 / k^2 \mu_+)$ when the fluids have greatly differing viscosities. These estimates of the growth rate agree with Hooper & Boyd's full asymptotic results for short wavelengths.

It should be noted that there is an implication in the above arguments that the out-of-phase components of vorticity are single-signed on each side of the interface midway between a peak and a trough. This implication is not true, as is readily seen when the upper fluid is much more viscous than the lower fluid. In this case the lower fluid sees an almost rigid interface, which will hence act as a source of counter-vorticity for the lower fluid. This counter-vorticity will not, however, reverse the sense of the circulation of the flow induced by the out-of-phase component of vorticity, because its purpose is only to reduce that flow in the lower fluid almost to rest at the nearly rigid interface.

Finally a remark must be made on the practical difficulties in observing this instability. If the fluids have no surface tension, then they will diffuse into one another with a diffusivity D . The condition that there is little diffusion over a wavelength before the instability has grown is

$$\left(\frac{\rho \gamma}{\mu k^2}\right)^2 \frac{\mu}{\rho D} > 1.$$

Thus the Schmidt number $\mu / \rho D$ has to be very large to offset the small Reynolds number $\rho \gamma / \mu k^2$. On the other hand, if the fluids are immiscible then the surface tension T will stabilize the disturbance with a decay rate $O(Tk / \mu_+)$. Thus the condition for surface tension to be negligible is

$$\frac{\rho \gamma^2}{k^3 T} > 1,$$

while the condition for the Reynolds number to be small is $\rho \gamma / \mu k^2 < 1$. These conflicting conditions are only satisfied by submicron wavelengths at shear rates of 10^9 s^{-1} for normal liquids with $\rho = 10^3 \text{ kg m}^{-3}$, $\mu = 10^{-3} \text{ N s m}^{-2}$ and $T = 3 \times 10^{-2} \text{ N m}^{-1}$. To have centimetre wavelengths at shear rates of one reciprocal second it is necessary to reduce the surface tension by a factor 3×10^{-5} .

REFERENCE

- HOOPER, A. P. & BOYD, W. G. C. 1983 *J. Fluid Mech.* **128**, 507.