Shocks

(1) Mass: $$\rho_1(V-u_1)=\rho_0 V$$

(2) Momentum: $$\rho_1(V-u_1)^2+p_1=\rho_0 V^2+p_0$$

(3) Bernoulli: $$\tfrac{1}{2}(V-u_1)^2+\frac{\gamma p_1}{(\gamma-1)\rho_1}=\tfrac{1}{2}V^2+\frac{\gamma p_0}{(\gamma-1)\rho_0}$$

Stage 1: use (1) and (2) to obtain velocities in terms of pressure and density.

Substitute $\rho_1(V-u_1)$ from (1) into (2),

$$\rho_0 V(V-u_1)+p_1=\rho_0 V^2+p_0,$$

i.e.

(4) $$\rho_0 V u_1=p_1-p_0.$$

But (1) gives $u_1=V(\rho_1-\rho_0)/\rho_1$ which can substitute into (4),

(5a) $$V^2=\frac{\rho_1}{\rho_0}\frac{p_1-p_0}{\rho_1-\rho_0}.$$

Substitute this into the right hand side of (1) squared,

(5b) $$(V-u_1)^2=\frac{\rho_0}{\rho_1}\frac{p_1-p_0}{\rho_1-\rho_0}.$$

And substitute (5a) into the left hand side (4) squared,

(5c) $$u_1^2=\frac{(\rho_1-\rho_0)(p_1-p_0)}{\rho_1\rho_0}.$$

Stage 2: use these results for the velocities in Bernoulli to obtain a relation between the pressures and densities.

Substitute (5a) and (5b) into (3),

$$\frac{\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1}-\frac{p_0}{\rho_0}\right)=\tfrac{1}{2}V^2-\tfrac{1}{2}(V-u_1)^2=\tfrac{1}{2}\frac{p_1-p_0}{\rho_1-\rho_0}\left(\frac{\rho_1}{\rho_0}-\frac{\rho_0}{\rho_1}=\frac{\rho_1^2-\rho_0^2}{\rho_1\rho_0}\right)$$

(6) $$=\tfrac{1}{2}(p_1-p_0)\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right),$$

the *Rankine–Hugoniot* result.

Stage 3: express above results in terms of pressure increase.

Define fractional pressure increase β, > 0 for shock traveling from side 1 to side 0,

$$p_1 = p_0(1+\beta).$$

Substituting into (6) gives

$$p_0 \frac{\gamma}{\gamma-1}\left(\frac{1+\beta}{\rho_1} - \frac{1}{\rho_0}\right) = p_0 \tfrac{1}{2}(1+\beta-1)\left(\frac{1}{\rho_0} - \frac{1}{\rho_1}\right),$$

i.e.

(7)
$$\frac{\rho_1}{\rho_0} = \frac{2\gamma+\gamma\beta+\beta}{2\gamma+\gamma\beta-\beta}$$

Note $\rho_1/\rho_0 < (p_1/p_0)^{1/\gamma}$, because entropy has increased.

Substituting (7) into (5a)

$$V^2 = \frac{\rho_1}{\rho_0}\frac{p_1-p_0}{\rho_1-\rho_0} = \frac{\rho_1}{\rho_0}\frac{p_0\beta}{\left(\frac{\rho_1}{\rho_0}-1\right)\rho_0} = \frac{2\gamma+\gamma\beta+\beta}{2\beta}\beta\left(\frac{p_0}{\rho_0} = \frac{1}{\gamma}c_0^2\right),$$

introducing the speed of sound c_0. Thus

(8)
$$\left(\frac{V}{c_0}\right)^2 = 1 + \frac{\gamma+1}{2\gamma}\beta > 1,$$

i.e. the gas approaches the shock supersonically.

Similarly (5b) can be re-expressed

$$(V-u_1)^2 = \frac{\rho_0}{\rho_1}\frac{p_1-p_0}{\rho_1-\rho_0} = \frac{p_1-p_0}{\left(\frac{\rho_1}{\rho_0}-1\right)\rho_1} = \frac{2\gamma+\gamma\beta-\beta}{2\beta}\frac{\beta}{1+\beta}\left(\frac{p_1}{\rho_1} = \frac{1}{\gamma}c_1^2\right).$$

Thus

(9)
$$\left(\frac{V-u_1}{c_1}\right)^2 = \frac{1+\frac{\gamma-1}{2\gamma}\beta}{1+\beta} < 1,$$

i.e. gas leaves the shock subsonically.