Small particles in a viscous fluid

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Small particles in a viscous fluid

Course in three parts

1. A quick course in micro-hydrodynamics
2. Sedimentation of particles
3. Rheology of suspensions

Good textbook for parts 1 & 2:
A Physical Introduction to Suspension Dynamics
by Elisabeth Guazzelli, Jeffrey F. Morris and Sylvie Pic
(Cambridge Texts in Applied Mathematics 2012).

A quick course in micro-hydrodynamics

Stokes equations

Simple properties

Flow past a sphere

More simple properties

Greens function

Effect of small inertia

A quick course in micro-hydrodynamics

Stokes equations
- Continuum mechanics
- Navier-Stokes equation
- Small Reynolds number
- Stokes equations

Simple properties

Flow past a sphere

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Continuum mechanics

Continuum description: mass density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$, etc.

Eulerian, not Lagrangian

Mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

But for constant density, flows are incompressible

$$\nabla\cdot\mathbf{u} = 0.$$

Forces in continuum description:

volume forces $\mathbf{F}$ and surface tractions $\sigma_{ij}n_j$.

So Cauchy momentum equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \nabla\cdot\boldsymbol{\sigma} + \mathbf{F}$$

Navier-Stokes equation

Newtonian viscous fluids

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},$$

with pressure $p(\mathbf{x}, t)$ determined globally by incompressibility

rather than locally by an equation of state $p(\rho)$

and strain-rate

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Note this is the most general relationship between σ and $\nabla\mathbf{u}$ which is linear, instantaneous and isotropic.

Hence the Navier-Stokes equation (momentum for a Newtonian viscous fluid) assuming μ constant.

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{F}$$

Boundary conditions

$$\mathbf{u}(\mathbf{x}) \quad \text{given, or} \quad \boldsymbol{\sigma}\cdot\mathbf{n}(\mathbf{x}) \quad \text{given}$$

Small Reynolds number

For a flow U over distances L, the Reynolds number is

$$Re = \frac{|\rho\mathbf{u}\cdot\nabla\mathbf{u}|}{|\mu\nabla^2\mathbf{u}|} = \frac{\rho U^2/L}{\mu U/L^2} = \frac{UL}{\nu}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

Small Reynolds number, $Re \ll 1$ if

- small U, e.g. $1\,\mathrm{cm/day}$ in oil reservoirs,
- small L, e.g. $10\,\mu\mathrm{m}$ bacteria,
- large ν, e.g. $10^8\,\mathrm{m^2/s}$ molten glass
- e.g. $1\,\mu\mathrm{m}$ water droplet falls under gravity in air at $0.1\,\mathrm{mm/s}$, so $Re = 10^{-5}$.

Stokes equations

If $Re \ll 1$, then Stokes flow (also called "creeping flow")

$$0 = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{F}$$
$$\text{with} \quad \nabla\cdot\mathbf{u} = 0.$$

Note: Stokes theory for $Re \ll 1$ usually works for $Re < 2$.

A quick course in micro-hydrodynamics

Stokes equations

Simple properties
- Linear and instantaneous
- Reversible in time
- Reversible in space

Flow past a sphere

More simple properties

Greens function

Effect of small inertia

Simple properties

1. Linear and Instantaneous
Dropping nonlinear $\mathbf{u}\cdot\nabla\mathbf{u}$ and time-dependent $\partial\mathbf{u}/\partial t$ leaves Stokes equations linear and instantaneous.

E.g. Rigid particle translation at $\mathbf{U}(t)$ in unbounded fluid
Flow $\mathbf{u}(\mathbf{x},t)$ linear & instantaneous in $\mathbf{U}(t)$, also $\boldsymbol{\sigma}(\mathbf{x},t)$, hence drag force

$$\mathbf{F}(t) = \mathbf{A}\cdot\mathbf{U}(t)$$

with $\mathbf{A}$ depending on size, shape, orientation and viscosity.

2. Reversible in time
Apply force $\mathbf{F}(t)$ in $0 \le t \le t_1$.
Now reverse force and its history, i.e. $\mathbf{F}(t) = -\mathbf{F}(2t_1 - t)$ in $t_1 \le t \le 2t_1$
Then flow $\mathbf{u}(\mathbf{x},t)$ and its history reverses.
Hence all fluid particles return to starting position.

Hence cannot swim at $Re \ll 1$ by reversible flapping.

Simple properties

3. Reversible in space:
Linearity + symmetry of geometry $\Rightarrow$ certain parts of $\mathbf{u}$ vanish.

E.g. 1. A sphere sedimenting next to a vertical wall does not migrate towards or away from the wall at $Re \ll 1$.

E.g. 2. Two equal spheres fall without separating. (Spin?)

E.g. 3. An ellipsoid (particle with three perpendicular planes of symmetry) falls under gravity without rotation in an unbounded flow.

E.g. 4. Two rigid spheres in a shear flow (possibly unequal, possibly next to a rigid wall) resume their original undisturbed streamlines after a collision.

A quick course in micro-hydrodynamics

Stokes equations

Simple properties

Flow past a sphere
- The solution
- Method 1
- Method 2
- Method 3
- Method 4
- Sedimenting sphere
- Rotating sphere
- Flow past an ellipsoid

More simple properties

Greens function

Flow past a sphere

Uniform flow **U** past a rigid sphere of radius a.

1. The solution – more important than derivation

$$\mathbf{u} = \mathbf{U}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(-\frac{3a}{4r^3} + \frac{3a^3}{4r^5}\right),$$

$$p = -\frac{3a\mu\mathbf{U}\cdot\mathbf{x}}{2r^3} \quad \text{and} \quad \boldsymbol{\sigma}\cdot\mathbf{n}|_{r=a} = \frac{3\mu}{2a}\mathbf{U}.$$

Hence the Stokes drag on the sphere is

$$\int_{r=a} \boldsymbol{\sigma}\cdot\mathbf{n}\, dS = 4\pi a^2 \frac{3\mu}{2a}\mathbf{U} = 6\pi\mu a\mathbf{U}.$$

Flow past a sphere

2. Solution method 1

The linearity of the Stokes equations means that $\mathbf{u}(\mathbf{x})$ must be linear in **U**.
Further, the problem has spherical symmetry about the centre of the sphere, which take as the origin.
The velocity and pressure fields must therefore take the forms

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{U}f(r) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})g(r), \\ p(\mathbf{x}) &= \mu(\mathbf{U}\cdot\mathbf{x})h(r), \end{aligned}$$

where $r = |\mathbf{x}|$, and f, g and h are functions of scalar r to be determined.

Now

$$\frac{\partial u_i}{\partial x_j} = U_i x_j f'/r + \delta_{ij} U_n x_n g + x_i U_j g + x_i x_j U_n x_n g'/r.$$

Flow past a sphere

solution method 1

Contracting i with j, we have the incompressibility condition

$$0 = \nabla\cdot\mathbf{u} = U_n x_n (f'/r + 4g + rg').$$

Differentiating again for momentum equation

$$\begin{aligned} \mu\nabla^2 u_i &= \mu U_i\left(f'' + 2f'/r + 2g\right) + \mu x_i U_n x_n \left(g'' + 6g'/r\right) \\ \nabla_i p &= \mu U_i h + \mu x_i U_n x_n h'/r \end{aligned}$$

Hence the governing equations give

$$f'/r + 4g + rg' = 0, \quad f'' + 2f'/r + 2g = h \quad \text{and} \quad g'' + 6g'/r = h'/r.$$

Eliminating h and then f yields

$$r^2 g''' + 11 r g'' + 24 g' = 0.$$

Solutions of the form $g = r^\alpha$.

Substituting, one finds $\alpha = 0$, -3 and -5, with associated $f = -(\alpha+4)r^{\alpha+2}/(\alpha+2)$ and $h = -(\alpha+5)(\alpha+2)r^\alpha$.

Flow past a sphere

solution method 1

Hence the general solution of the assumed form linear in **u** is

$$\mathbf{u}(\mathbf{x}) = \mathbf{U}\left(-2Ar^2 + B + Cr^{-1} - \tfrac{1}{3}Dr^{-3}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(A + Cr^{-3} + Dr^{-5}\right),$$

$$p(\mathbf{x}) = \mu(\mathbf{U}\cdot\mathbf{x})\left(-10A + 2Cr^{-3}\right).$$

We shall need the stress exerted across a spherical surface with unit normal $\mathbf{n} = \mathbf{x}/r$

$$\boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{U}\left(-3Ar + 2Dr^{-4}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(9Ar^{-1} - 6Cr^{-4} - 6Dr^{-6}\right)$$

Applying the boundary conditions on the rigid sphere and for the far field, we find the coefficients

$$A = 0, \qquad B = 1, \qquad C = -\tfrac{3}{4}a \quad \text{and} \quad D = \tfrac{3}{4}a^3$$

For solution given earlier

Flow past a sphere

2. Solution method 2

Use a Stokes streamfunction for the axisymmetric flow

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\theta} \quad \text{and} \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}.$$

The vorticity equation (curl of the momentum equation, to eliminate the pressure) is then at low Reynolds numbers

$$\mathcal{D}^2\mathcal{D}^2\Psi = 0 \quad \text{where} \quad \mathcal{D}^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

The uniform flow at infinity has $\Psi = \frac{1}{2}Ur^2\sin^2\theta$, so one tries $\Psi = F(r)\sin^2\theta$, and finds $F = Ar^4 + Br^2 + Cr + D/r$.

Flow past a sphere

3. Solution method 3

One can show (Papkovich-Neuber) that the general solution of the Stokes equation can be expressed in terms of a vector harmonic function $\boldsymbol{\phi}(\mathbf{x})$ (i.e. $\nabla^2\boldsymbol{\phi} = 0$)

$$\mathbf{u} = 2\boldsymbol{\phi} - \nabla(\mathbf{x}\cdot\boldsymbol{\phi}) \quad p = -2\mu\nabla\cdot\boldsymbol{\phi}.$$

$$\sigma_{ij} = 2\mu\left(\delta_{ij}\frac{\partial\phi_n}{\partial x_n} - x_k\frac{\partial^2\phi_k}{\partial x_i\partial x_j}\right).$$

Linearity and spherical symmetry then give

$$\boldsymbol{\phi} = A\mathbf{U}\frac{1}{r} + B\mathbf{U}\cdot\nabla\nabla\frac{1}{r},$$

with coefficients A and B to be determined by applying the boundary conditions.

Flow past a sphere

4. Solution method 4

The pressure and vorticity are harmonic functions.

Using linearity and spherical symmetry, they must take the form

$$p = \mu A\mathbf{U}\cdot\mathbf{x}/r^3 \quad \text{and} \quad \nabla\wedge\mathbf{u} = B\mathbf{U}\wedge\mathbf{x}/r^3.$$

The final step to $\mathbf{u}$ is tedious.

Sedimentation of a rigid sphere

Force balance, with densities ρ_s of sphere and ρ_f of fluid

0	$=$	$\rho_s\frac{4\pi}{3}a^3g$	$-$	$\rho_f\frac{4\pi}{3}a^3g$	$-$	$6\pi\mu a\mathbf{U}$
no inertia		weight		buoyancy		Stokes drag

So Stokes settling velocity

$$\mathbf{U} = \frac{2\Delta\rho a^2 g}{9\mu}$$

E.g. $1\,\mu\mathrm{m}$ sphere, $\Delta\rho = 10^3\,\mathrm{kg\,m^{-3}}$, water $\mu = 10^{-3}\,\mathrm{Pa\,s}$ gives $U = 2\mu\mathrm{m/s}$, i.e. falls through diameter in a second. (Check $Re = 10^{-6}$)

Drag on a fluid sphere (Student exercise!)

$$\mathbf{F} = -2\pi\frac{2\mu_f+3\mu_s}{\mu_f+\mu_s}\mu_f aU$$

Rotation of a rigid sphere

Sphere rotating at angular velocity $\boldsymbol{\Omega}$.
Flow

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\Omega} \wedge \mathbf{x}\frac{a^3}{r^3}$$

(A potential flow, so satisfies Stokes equations.)

Hence couple on sphere – student exercise!

$$\mathbf{G} = 8\pi\mu a^3 \boldsymbol{\Omega}$$

Stokes flow past an ellipsoid

For principle semi-diameters a_1, a_2, a_3 Oberbeck (1876) found

$$\text{Force} \quad F_1 = -\frac{16\pi\mu U_1}{L + a_1^2 K_2} \quad \text{and Couple} \quad G_1 = -\frac{16\pi\mu(a_2^2 + a_3^2)}{3(a_2^2 K_2 + a_3^2 K_3)}$$

where

$$L = \int_0^\infty \frac{d\lambda}{\Delta(\lambda)} \quad \text{and} \quad K_i = \int_0^\infty \frac{d\lambda}{(a_i^2 + \lambda)\Delta(\lambda)}$$

with $\Delta^2 = (a_1^2 + \lambda)(a_3^2 + \lambda)(a_3^2 + \lambda)$.

For a disk $a_1 \ll a_2 = a_3$

$$F_1 \sim 16\pi\mu a_2 U_1, \quad F_2 \sim \tfrac{32}{3}\mu a_2 U_2, \quad G_i \sim \tfrac{8}{3}\mu a_2 \Omega_i$$

For a rod $a_1 \gg a_2 = a_3$, where $\ln = \ln \frac{2a_1}{a_2}$

$$F_1 \sim \frac{4\pi\mu a_1 U_1}{\ln - \frac{1}{2}}, \quad F_2 \sim \frac{8\pi\mu a_1 U_2}{\ln + \frac{1}{2}}, \quad G_1 \sim \tfrac{16}{3}\pi\mu a_1 a_2 \Omega_1, \quad G_2 \sim \frac{\frac{8}{3}\pi\mu a_1^3 \Omega_2}{\ln - \frac{1}{2}}$$

Important conclusion Drag $\approx 6\pi\mu$ with largest diameter.

A quick course in micro-hydrodynamics

Stokes equations

Simple properties

Flow past a sphere

More simple properties
- A useful result
- Minimum dissipation
- Uniqueness
- Geometric bounding
- Reciprocal theorem
- Symmetry of resistance matrix
- Faxen's formula

Greens function

More simple properties

1. A useful result

Let $\mathbf{u}^S(\mathbf{x})$ be a Stokes flow with $\mathbf{F} = 0$ in V, and let $\mathbf{u}(\mathbf{x})$ be any other incompressible flow, then

$$\int_V 2\mu e_{ij}^S e_{ij}\, dV = \int_S \sigma_{ij}^S n_j u_i\, dA.$$

Because

$$2\mu e_{ij}^S = \sigma_{ij}^S + p^S \delta_{ij} \quad \text{and} \quad p^S \delta_{ij} e_{ij} = p^S \nabla\cdot\mathbf{u} = 0 \qquad (1)$$

$$\text{so} \quad 2\mu e_{ij}^S e_{ij} = \sigma_{ij}^S e_{ij}. \qquad (2)$$

And

$$\sigma_{ij}^S = \sigma_{ji}^S \quad \text{so} \qquad (3)$$

$$\sigma_{ij}^S e_{ij} = \sigma_{ij}^S \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\sigma^S ij u_i\right) - \underbrace{\frac{\partial \sigma_{ij}^S}{\partial x_j}}_{=\mathbf{F}=0} u_i \qquad (4)$$

Hence result by divergence theorem.

More simple properties
2. Minimum dissipation

Let $\mathbf{u}(\mathbf{x})$ and $\mathbf{u}^S(\mathbf{x})$ be two incompressible flows in V, both satisfying the same boundary condition $u = u^S = \mathbf{U}(\mathbf{x})$ give on S. Let $\mathbf{u}^S$ also satisfy the Stokes equation with $\mathbf{F} = 0$ in V.
Then

$$\int_V 2\mu e_{ij} e_{ij}\, dV = \int_V 2\mu e^S_{ij} e^S_{ij}\, dV$$
$$+ \underbrace{\int_V 2\mu(e_{ij} - e^S_{ij})(e_{ij} - e^S_{ij})\, dV}_{\longleftarrow\text{positive}\longrightarrow} + \int_V 4\mu e^S_{ij}(e_{ij} - e^S_{ij})\, dV.$$

The last integral is of the form of the useful result

$$\int_V 4\mu e^S_{ij}(e_{ij} - e^S_{ij})\, dV = \int_S 2\sigma^S_{ij} n_j (u_i - u^S_i)\, dA = 0 \quad \text{by bc}$$

More simple properties
2. Minimum dissipation

Hence

$$\int_V 2\mu e_{ij} e_{ij}\, dV \geq \int_V 2\mu e^S_{ij} e^S_{ij}\, dV,$$

i.e. the Stokes flow $\mathbf{u}^S(\mathbf{x})$ has the minimum dissipation out of all incompressible flows satisfying the boundary condition

Hence e.g. drag larger at non-zero Reynolds number.

Warning: Same geometry. Cannot select geometry by minimum dissipation.

More simple properties
3. Uniqueness

If $\mathbf{u}^1(\mathbf{x})$ and $\mathbf{u}^2(\mathbf{x})$ are two Stokes flows in V satisfying the same boundary conditions, then minimum dissipation gives

$$\int_V 2\mu(e^1_{ij} - e^2_{ij})(e^1_{ij} - e^2_{ij})\, dV = 0$$

Hence

$$e^1_{ij} - e^2_{ij} = 0 \quad \text{in } V,$$

i.e. $\mathbf{u}^1 - \mathbf{u}^2$ is strainless, i.e. a solid body translation + rotation, i.e. zero by the boundary conditions. Hence

$$\mathbf{u}^1(\mathbf{x}) = \mathbf{u}^2(\mathbf{x}) \quad \text{in } V.$$

Hence Stokes flows are unique.

More simple properties
4. Geometric bounding

Rigid cube, sides of length $2L$, moving at $\mathbf{U}$, drag $\mathbf{F}$.
Let $\mathbf{u}^S(\mathbf{x})$ be Stokes flow outside cube, V.
Then dissipation

$$\int_V 2\mu e^S_{ij} e^S_{ij}\, dV = \text{rate of working by surface forces} = -\mathbf{U}\cdot\mathbf{F}.$$

Cube just contained by sphere radius $a = \sqrt{3}L$, also moving at $\mathbf{U}$.

Define second flow

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \text{the Stokes flow for sphere} & \text{outside sphere,} \\ \mathbf{U} & \text{in gap.} \end{cases}$$

More simple properties

4. Geometric bounding

For this second flow

$$\int_V 2\mu e_{ij}e_{ij}\,dV = \int_{r>a} 2\mu e_{ij}e_{ij}\,dV \quad \text{because } e=0 \text{ in gap,}$$
$$= \text{rate of working by sphere} = 6\pi\mu\sqrt{3}L\mathbf{U}\cdot\mathbf{U}$$

Hence minimum dissipation bounds drag $\mathbf{F}$ on cube

$$-\mathbf{F}\cdot\mathbf{U} \le 6\pi\mu\sqrt{3}L\mathbf{U}\cdot\mathbf{U}$$

Similarly for sphere just contained inside cube

$$6\pi\mu L\mathbf{U}\cdot\mathbf{U} \le -\mathbf{F}\cdot\mathbf{U}$$

Student excercise: bound for tetrahedron (not so tight).

More simple properties

5. Reciprocal theorem

For the same volume V, let $\mathbf{u}_1$ be the Stokes flow with volume forces $\mathbf{f}_1$ satisfying boundary conditions $\mathbf{u}_1 = \mathbf{U}_1$.
Let $\mathbf{u}_2$ be Stokes flow in V with $\mathbf{f}_2$ and $\mathbf{U}_2$.

Then by the useful result

$$\int_V \mathbf{u}_1\cdot\mathbf{f}_2\,dV + \int_S \mathbf{U}_1\cdot\boldsymbol{\sigma}_2\cdot\mathbf{n}\,dA = \int_V 2\mu\mathbf{e}_1 : \mathbf{e}_2\,dV$$
$$= \int_V \mathbf{u}_2\cdot\mathbf{f}_1\,dV + \int_S \mathbf{U}_2\cdot\boldsymbol{\sigma}_1\cdot\mathbf{n}\,dA$$

i.e. work done by one velocity on the forces of the other is vice versa.
Greens theorem in any other subject

More simple properties

6. Reciprocal theorem – application to resistance matrix

General rigid body motion in fluid at rest a infinity, translating at $\mathbf{U}(t)$ and rotating (about a selected point) $\boldsymbol{\Omega}(t)$.
By linearity and instantaneity, the force $\mathbf{F}(t)$ and couple $\mathbf{G}(t)$ (about the same selected point)

$$\begin{pmatrix}\mathbf{F}\\ \mathbf{G}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{U}\\ \boldsymbol{\Omega}\end{pmatrix}$$

The Reciprocal theorem gives for the two rigid body motions

$$\mathbf{U}_1\cdot\mathbf{F}_2 + \boldsymbol{\Omega}_1\cdot\mathbf{G}_2 = \mathbf{U}_2\cdot\mathbf{F}_1 + \boldsymbol{\Omega}_2\cdot\mathbf{G}_1$$

True all $\mathbf{U}_1$ etc, so

$$\mathbf{A} = \mathbf{A}^T, \quad \mathbf{B} = \mathbf{C}^T \quad \text{and} \quad \mathbf{D} = \mathbf{D}^T.$$

More simple properties

6. Reciprocal theorem – application to resistance matrix

$\mathbf{B} = \mathbf{C}^T$ means

force due to rotation = couple due to translating.

$\mathbf{A} = \mathbf{A}^T$ & $\mathbf{D} = \mathbf{D}^T$ for symmetric cube means

$$\mathbf{A}\ \&\ \mathbf{D} \quad \text{diagonal, and} \quad \mathbf{B} = \mathbf{C}^T = 0$$

Thus drag on a cube is parallel to velocity,
also for symmetric tetrahedron.

Need "corkscrew" feature for $\mathbf{B} \ne 0$.

More simple properties
7. Reciprocal theorem – application to Faxen's formula

A force-free sphere placed in arbitrary flow $\mathbf{u}^\infty(\mathbf{x})$ moves with what velocity $\mathbf{V}$? Forces which generate $\mathbf{u}^\infty$ kept constant.

Let $\mathbf{u}^+$ be Stokes flow with sphere inserted in $\mathbf{u}^\infty$.
Then disturbance flow is $\mathbf{u}_1 = \mathbf{u}^+ - \mathbf{u}^\infty$.
Now same forces for $\mathbf{u}^+$ & $\mathbf{u}^\infty$, so $\mathbf{f}_1 = 0$. Also $\mathbf{u}_1 \to 0$ far from sphere.
Let $\mathbf{u}_2(\mathbf{x})$ be flow outside a sphere translating at $\mathbf{U}_2$ with $\mathbf{f}_2 = 0$.

Applying Reciprocal theorem

$$\int \mathbf{u}_1 \cdot \underbrace{\mathbf{f}_2}_{=0} \, dV + \int \mathbf{u}_1 \cdot \underbrace{\boldsymbol{\sigma}_2 \cdot \mathbf{n}}_{=-3\mu\mathbf{u}_2/a} \, dA = \int \mathbf{u}_2 \cdot \underbrace{\mathbf{f}_1}_{=0} \, dV + \int \underbrace{\mathbf{u}_2}_{=\mathbf{U}_2} \cdot \boldsymbol{\sigma}_1 \cdot \mathbf{n} \, dA.$$

Now look at RHS and then LHS, using $\mathbf{u}_1 = \mathbf{u}^+ - \mathbf{u}^\infty$.

More simple properties
7. Reciprocal theorem – application to Faxen's formula

$$\mathrm{RHS} = \mathbf{U}_2 \cdot \left(\int \boldsymbol{\sigma}^+ \cdot \mathbf{n} \, dA - \int \boldsymbol{\sigma}^\infty \cdot \mathbf{n} \, dA = 0 - 0 \right) = 0,$$

as both integrals are force on sphere.
Hence

$$\mathrm{LHS} = -\frac{3\mu}{a} \mathbf{U}_2 \cdot \left(\int \underbrace{\mathbf{u}^+}_{=\mathbf{V}} \, dA - \int \mathbf{u}^\infty \, dA \right) = \mathrm{RHS} = 0$$

For all $\mathbf{U}_2$, so velocity of sphere inserted into $\mathbf{u}^\infty(\mathbf{x})$ is

$$\mathbf{V} = \frac{1}{4\pi a^2} \int_{r=a} \mathbf{u}^\infty(\mathbf{x}) \, dA$$

More simple properties
7. Reciprocal theorem – application to Faxen's formula

$$\mathbf{V} = \frac{1}{4\pi a^2} \int_{r=a} \mathbf{u}^\infty(\mathbf{x}) \, dA$$

Finally use a Taylor series

$$\mathbf{u}^\infty(\mathbf{x}) = \mathbf{u}^\infty(0) + \mathbf{x} \cdot \nabla \mathbf{u}^\infty|_0 + \tfrac{1}{2}\mathbf{x}\mathbf{x} : \nabla\nabla\mathbf{u}^\infty|_0 + \cdots$$

Integrating over the sphere, the odd terms vanish by symmetry, so

$$\mathbf{V} = \mathbf{u}^\infty(0) + \tfrac{a^2}{6} \nabla^2 \mathbf{u}^\infty|_0$$

with higher even terms vanishing by ∇^{2n}(Stokes equations) $= 0$.

A quick course in micro-hydrodynamics

Stokes equations

Simple properties

Flow past a sphere

More simple properties

Greens function
- Stokeslet
- Integral representation
- Slender-body theory

Effect of small inertia

Greens function for Stokes equations
or 'Stokeslet'

For a point momentum source

$$\nabla\cdot\mathbf{u} = 0$$
$$0 = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{F}\delta(\mathbf{x})$$

Solution – more important than derivation

$$\mathbf{u}(\mathbf{x}) = \mathbf{F}\cdot\mathbf{G}(\mathbf{x}) = \frac{1}{8\pi\mu}\left(\mathbf{F}\frac{1}{r} + (\mathbf{F}\cdot\mathbf{x})\mathbf{x}\frac{1}{r^3}\right)$$
$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{F}\cdot\mathbf{K}(\mathbf{x}) = -\frac{3}{4\pi}\mathbf{F}\cdot\mathbf{xxx}\frac{1}{r^5}$$

$\mathbf{G}$ is called the 'Oseen tensor'.

Already seen this in flow past a sphere:

Greens function for Stokes equations
Far field for a sphere

Far from the sphere $r \gg a$, the flow is

$$\mathbf{u} = \mathbf{U}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(-\frac{3a}{4r^3} + \frac{3a^3}{4r^5}\right),$$

But the drag is $\mathbf{F} = -6\pi\mu a\mathbf{U}$, i.e.

$$\mathbf{u} - \mathbf{U} \sim \frac{1}{8\pi\mu}\left(\mathbf{F}\frac{1}{r} + (\mathbf{F}\cdot\mathbf{x})\mathbf{x}\frac{1}{r^3}\right)$$

Hence far-field due to force is universal, independent of particle shape.

Greens function for Stokes equations
Integral representation

To solve

$$\nabla\cdot\mathbf{u} = 0$$
$$0 = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}(\mathbf{x})$$

with boundary conditions on $\mathbf{u}(\mathbf{x})$ or $\boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}$.

Use the Reciprocal theorem (Greens theorem) with $\mathbf{u}_1$ for the unknown flow and $\mathbf{u}_2$ for the Greens function for point source at $\mathbf{x}'$

$$\mathbf{u}(\mathbf{x}') = \underbrace{\int_V \mathbf{G}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{f}\,dV}_{\text{forces in } V}$$
$$+ \int_S \Big(\underbrace{\mathbf{G}(\mathbf{x}-\mathbf{x}')\cdot\boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}}_{\text{forces on } S} - \underbrace{(\mathbf{K}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{n})\cdot\mathbf{u}}_{\text{dipoles on } S}\Big)\,dA$$

Greens function for Stokes equations
Integral representation – Boundary integral Method

Letting $\mathbf{x}'$ in V tend onto the surface S yields an integral equation for the unknown $\mathbf{u}$ (or $\boldsymbol{\sigma}\cdot\mathbf{n}$) on S in terms of the known $\boldsymbol{\sigma}\cdot\mathbf{n}$ (or $\mathbf{u}$) on S.

Delicate limit $\mathbf{x}' \to S$: $\int \mathbf{K}\cdot\mathbf{n} \to +\frac{1}{2}\mathbf{u}$ for $\mathbf{x}'$ in V, $-\frac{1}{2}\mathbf{u}$ for $\mathbf{x}'$ outside V, more complex at corners on S.

Basis of numerical *Boundary Integral Method*.

Advantage: fewer points on surface than in volume, and no infinity.

Disadvantages: Special attention needed in numerical evaluation of singular integrals, and there are often eigensolutions, e.g. constant pressure induces no flow.

Greens function for Stokes equations

Integral representation - for a suspended drop

Extension to a drop of viscosity $\lambda\mu$ surrounded by a fluid of viscosity μ.

One knows the jump across the interface of the normal viscous stress

$$\boldsymbol{\sigma}\cdot\mathbf{n} = \gamma\kappa\mathbf{n}$$

with surface tension γ and surface curvature $\mathbf{n}$.

Adding λ times the integral equation for the interior to that for the exterior, one reduce the stress contribution to its difference

$$\tfrac{1}{2}(1+\lambda)\mathbf{u}(\mathbf{x}') = \mathbf{u}^{\infty}(\mathbf{x}')$$
$$-\int_S \mathbf{G}(\mathbf{x}-\mathbf{x}')\cdot\gamma\kappa\mathbf{n}\,dA - (1-\lambda)\int_S \mathbf{K}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{n}\cdot\mathbf{u}\,dA$$

Greens function for Stokes equations

Slender-body theory

For slender bodies, approximate the distribution of forces $\boldsymbol{\sigma}\cdot\mathbf{n}$ on the surface S by a distribution of forces $\mathbf{f}(s)$ along the centreline $\mathbf{X}(s)$ in $-L \le s \le L$

$$\mathbf{u}(\mathbf{x}') = \mathbf{u}^{\infty}(\mathbf{x}') + \int_{-L}^{L} \mathbf{G}(\mathbf{X}(s)-\mathbf{x}')\cdot\mathbf{f}(s)\,ds$$

Satisfy the boundary condition by evaluating at distance $\epsilon R(s_0)$ from centreline at $s = s_0$, either numerically

or asymptotically for

$$\mathbf{f}(s_0) \sim \frac{2\pi\mu}{\ln\frac{L}{R}}\left(2\mathbf{I} - \mathbf{X}'\mathbf{X}'\right)\cdot\left(\mathbf{U}(s_0) - \mathbf{u}^{\infty}(\mathbf{X}(s_0))\right)$$

Effects of small inertia on flow past a sphere

Whitehead paradox

Flow $\mathbf{U}$ past a sphere of radius a.
At $Re \ll 1$, first approximation is given by Stokes flow.
Attempt to find correction as regular perturbation fails.
In far field

$$u = U + O\left(\frac{Ua}{r}\right), \quad \text{so} \quad \rho\mathbf{u}\cdot\nabla\mathbf{u} = O\left(\frac{\rho U^2 a}{r^2}\right)$$

A correction $\mathbf{u}_2$ forced by this

$$-\nabla p_2 + \mu\nabla^2\mathbf{u}_2 = \rho\mathbf{u}\cdot\nabla\mathbf{u}|_{\text{Stokes}}$$

would give

$$u_2 = O(\rho U^2 a/\mu)$$

which does not decay.

Effects of small inertia on flow past a sphere

Oseen equation

Look more carefully at Stokes solution. In far field

$$\text{Viscous terms} = O\left(\frac{\mu U a}{r^3}\right), \quad \text{while interial terms} = O(\left(\frac{\rho U^2 a}{r^2}\right)$$

so at $r = \mu/U$ inertial terms no longer small.

Fortunately in far field $\mathbf{u} = \mathbf{U} + \mathbf{u}'$ with $\mathbf{u}'$ small, so can linearise Navier-Stokes

$$\rho\mathbf{U}\cdot\nabla\mathbf{u}' = -\nabla p' + \mu\nabla^2\mathbf{u}' + \mathbf{F}\delta(\mathbf{x})$$

where in far field sphere appears a point force.

Solve by Fourier transforms or representation

$$\mathbf{u}' = \nabla\phi + \nu\nabla\chi - \mathbf{U}\chi \quad \text{and} \quad p' = -\rho\mathbf{U}\cdot\nabla\phi.$$

Effects of small inertia on flow past a sphere
Oseen equation solved

Find

$$\phi = -\frac{3a\nu}{2r}\text{(point volume source)} \quad \text{and} \quad \chi = \frac{3a}{2r} e^{\left(\frac{\mathbf{U}\mathbf{x}}{2\nu} - \frac{Ur}{2\nu}\right)}$$

Look nearer to sphere $a \ll r \ll \nu/U$, $\quad r^{-2}$ terms cancel and

$$\mathbf{u}' \sim \frac{1}{r}\ \text{Stokeslet} + \mathbf{U}\frac{3Ua}{4\nu}\ \text{uniform flow}$$

Hence drag increases by $1 + \frac{3}{8}Re$.

Effects of small inertia on flow past a sphere
Oseen wake

Look far from sphere $r \gg \nu/U$ near downstream axis

$$\mathbf{u}' \sim -\mathbf{U}\frac{3a}{2z} e^{-\frac{U(x^2+y^2)}{4\nu z}}$$

i.e. a wake diffusing to $r = \sqrt{\nu(t = z/U)}$ with mass flux deficit

$$\int \rho u'\, dxdy = -6\pi\mu Ua \quad = \text{momentum deficit } /U$$

Helpful idea at higher Re

Impacts on time-dependent flows – Basset history wrong

Missing mass flux in wake goes to point source ϕ-flow.

A quick course in micro-hydrodynamics

Stokes equations

Simple properties

Flow past a sphere

More simple properties

Greens function

Effect of small inertia