

**Example Sheet 1: Sound Waves**

1. *Plane waves and radiation.* A thin piston executes very small oscillations about  $x = 0$  in a long straight fluid-filled tube with cross-sectional area  $A$  and rigid walls aligned with the  $x$ -axis. Given the piston velocity  $\dot{X}(t)$ , find the velocity potential  $\phi(x, t)$  for the (linearised) sound waves generated in  $x > 0$  and  $x < 0$  (linearising  $X \approx 0$ ). Show that if  $x_0 > 0$  the total power  $AI_x$  radiated across  $x = x_0$  is

$$(q(t - x_0/c_0))^2 c_0 / \rho_0 A, \quad (1)$$

where  $q(t) = \rho_0 A \dot{X}(t)$  is the rate at which mass is displaced on one side of the piston. What is the corresponding result for  $x_0 < 0$ ? [*Later in the course, we will analyse the effects of nonlinearity.*]

2. *Reflection and transmission.* An interface at  $x = 0$  separates fluid of density  $\rho_0$  and sound speed  $c_0$  in  $x < 0$  from fluid of density  $\rho_1$  and sound speed  $c_1$  in  $x > 0$ . A plane harmonic sound wave is incident from  $x < 0$  with wavevector  $\mathbf{k} = (k, 0, 0)$  and amplitude  $A$  (of its pressure perturbation). What is the frequency  $\omega$  and the wavevector  $\mathbf{k}'$  of the transmitted sound wave in  $x > 0$ ?

Write down the form of the pressure perturbation in  $x < 0$  and  $x > 0$ , find the corresponding velocity potential and state the interfacial boundary conditions. Hence find the amplitudes of the reflected and transmitted waves.

Assume wlog that  $A = 1$ . Verify that the time-averaged acoustic energy flux is conserved. When is all the energy flux transmitted? How much is reflected if  $\rho_0 \gg \rho_1$  and  $c_0 \approx c_1$ ?

3. *Evanescence waves near an interface.* Find solutions to the wave equation of the form

$$\phi(x, y, t) = \exp(ikx - i\omega t)f(y), \quad (2)$$

where  $k > \omega/c_0 > 0$ . Hence find the solution in  $y \geq 0$  in which there is no disturbance as  $y \rightarrow \infty$  and waves are forced by the inhomogeneous boundary condition

$$v = \text{Re} [v_0 \exp(ikx - i\omega t)] \quad \text{on} \quad y = 0.$$

Here  $\nabla\phi = (u, v, 0)$  and  $v_0$  is a real constant. Over what lengthscale do the waves decay away from the boundary?

Calculate the time-averaged acoustic energy flux  $\langle \mathbf{I} \rangle$  and verify that:

(a) the energy flux perpendicular to the boundary  $y = 0$  satisfies  $\langle I_y \rangle = 0$ ;

(b) the energy flux parallel to the boundary satisfies  $\langle I_x \rangle = c \langle E \rangle$  at any position  $y$ , where  $E$  is the acoustic energy density and  $c = \omega/k$  is the phase velocity in the  $x$ -direction. [*Since  $c < c_0$ , the disturbance and its energy travel subsonically along the boundary.*]

\*Assuming that surface tension and gravity are negligible, determine whether a non-zero solution can exist in which evanescent sound waves propagate along both sides of a (force-free) interface between two fluids with different physical properties in  $y < 0$  and  $y > 0$ ,

4. *Acoustic waveguide.* Find solutions to the wave equation of the form (2) for a region  $0 < y < h$  with a rigid boundary at  $y = 0$  and a free boundary at  $y = h$ . (Take  $\omega > 0$ , but make no *a priori* assumption about  $k$ .) Show that a wave can propagate in the  $x$ -direction only if  $\omega$  exceeds a critical value  $\omega_c$ . What happens if a disturbance is generated at  $x = 0$  with frequency  $\omega < \omega_c$ ?

5. *Spherical waves and radiation.* Explain why the general spherically symmetric solution  $\phi(r, t)$  to the wave equation can be written as

$$\phi = \frac{-1}{4\pi\rho_0} \left( \frac{q(t - r/c_0)}{r} + \frac{Q(t + r/c_0)}{r} \right), \quad (3)$$

where  $q$  and  $Q$  are arbitrary functions. Assume from now on that there are only outgoing waves. Calculate the radial velocity  $u_r$  and the pressure perturbation  $\tilde{p}$ .

(a) By considering the volume flux through a sphere of radius  $\epsilon$  as  $\epsilon \rightarrow 0$ , show that  $q(t)$  is the mass flux out of  $r = 0$ . Show also that  $\phi$  actually satisfies

$$\nabla^2 \phi - c_0^{-2} \partial^2 \phi / \partial t^2 = q(t) \delta(\mathbf{x}) / \rho_0, \quad (4)$$

where  $\delta$  is the Dirac delta function. (*Hint:* integrate (4) over  $r \leq \epsilon$  and let  $\epsilon \rightarrow 0$ .) [*The notation in (3) is standard and motivated by the meaning of  $q$ . In the detailed calculations below, it may be easier to write  $q/4\pi\rho_0 = f$ .*]

(b) Show that in the far-field, i.e. for ‘large’  $r$ , the kinetic energy density  $K$ , the potential energy density  $W$ , and the acoustic-energy flux  $\mathbf{I} = \tilde{p}\mathbf{u}$ , approximately satisfy the same equations,  $K = W$  and  $I = (K + W)c_0$ , as in a plane wave. Similarly, show that the total power radiated across a ‘large’ sphere of radius  $R$  is approximately

$$(\dot{q}(t - R/c_0))^2 / 4\pi\rho_0 c_0. \quad (5)$$

[*The solution (3) with  $Q = 0$  is called a point source, or an acoustic monopole of strength  $\dot{q}(t)$ .*]

What does ‘large  $r$ ’ mean for a time-harmonic source with  $q(t) = \text{Re}(q_0 e^{i\omega t})$ ?

(c) For the same time-harmonic source, show that  $\langle K \rangle / \langle W \rangle \sim (c_0/r\omega)^2$  as  $r \rightarrow 0$  and find both  $I_r$  and  $\langle I_r \rangle$  in the same limit. Comment on these results. Compare (1) with (5) for the case  $A \ll (c_0/\omega)^2$ . (What does this condition mean physically?) [*This is one of the principles behind the ‘horn loudspeaker’.*]

6. *Harmonic series.* Explain why (3) describes a possible acoustic disturbance in a conical tube of any cross-sectional shape. Model an oboe (with all the finger-holes closed) as a small-angle conical tube of length  $\ell$ : at the narrow end the cross-sectional area is effectively zero and  $\tilde{p}$  is finite; the larger end is open and  $\tilde{p}$  may be assumed to be zero. [*This is a good approximation only if the radius of the larger end is much less than  $c_0/\omega$ .*] Show that the instrument has a set of normal modes (i.e. standing-wave solutions of the form  $R(r)e^{-i\omega t}$ ) with frequencies

$$\omega\ell/c_0 = n\pi \quad (n \in \mathbb{Z}). \quad (\dagger)$$

If, instead, the larger end is closed, so that the radial velocity is zero there, show that the corresponding normal-mode frequencies are the solutions of

$$\omega\ell/c_0 = \tan(\omega\ell/c_0). \quad (\ddagger)$$

Find approximate solutions of  $(\ddagger)$  in the high-frequency limit. [*The set of frequencies  $(\dagger)$  forms a musical ‘harmonic series’, while the set  $(\ddagger)$  does not.*]

7. *An oscillating bubble* (Tripos 93124). A bubble makes small spherically symmetric oscillations in a compressible inviscid fluid. When the radius  $a(t)$  is perturbed slightly from its mean value  $a_0$ , the internal dynamics of the bubble produces a pressure  $-\kappa(a - a_0)$  on the bubble surface. Derive the linearised equation of motion for the oscillations

$$\rho_0 a_0 \ddot{a} + \frac{\kappa a_0}{c_0} \dot{a} + \kappa(a - a_0) = 0,$$

where  $\rho_0$  is the undisturbed density of the fluid and  $c_0$  is the sound speed (you may quote results from question 5). What is the mechanism of energy loss from the oscillations represented by the ‘damping’ term in this ODE for  $a$ ?

8\*. *Images*. Explain briefly how the method of images can be used to find the sound field produced by a point source placed either near a plane rigid boundary, or in the corner between two plane rigid boundaries at right-angles (i.e. find an image system that satisfies the boundary conditions).

For each case, use the results of 5(b) to write down an approximation to the time-averaged total power radiated by a time-harmonic point source if the distance of the source to the boundaries is much less than a wavelength. Will a whistle sound louder if blown near a wall?

9\*. *Source with boundaries*. A point source (monopole) is placed at  $\mathbf{x} = \mathbf{x}_0$  inside a straight semi-infinite tube aligned along the positive  $x$ -axis with a closed end at  $x = 0$  and cross-sectional area  $A$ . By integrating (4) with  $\delta(\mathbf{x})$  replaced by  $\delta(\mathbf{x} - \mathbf{x}_0)$ , and using the boundary conditions, show that the cross-sectional average potential

$$\bar{\phi}(x, t) = \frac{1}{A} \int \int \phi \, dy \, dz$$

satisfies

$$\frac{\partial^2 \bar{\phi}}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 \bar{\phi}}{\partial t^2} = \frac{q(t)}{\rho_0 A} \delta(x - x_0).$$

[*Hint*: Recall  $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$ .] What conditions should be imposed on  $\bar{\phi}$  and  $\partial\bar{\phi}/\partial x$  at  $x = 0$ ,  $x = x_0$  and as  $x \rightarrow \infty$ ?

For a time-harmonic source,  $q(t) = q_0 e^{i\omega t}$  (real part understood), show that

$$\bar{\phi} = \frac{ic_0 q_0}{\omega \rho_0 A} \frac{\exp\{i\omega[t - (x - x_0)/c_0]\}}{1 + i \tan(\omega x_0/c_0)} \quad \text{in } x > x_0.$$

If  $A \ll (c_0/\omega)^2$ , why is it reasonable to assume that the sound field is almost one-dimensional (i.e.  $\phi(\mathbf{x}, t) \approx \bar{\phi}(x, t)$ ) except near  $x = x_0$ ? Making this assumption, show that if  $x_0 \ll c_0/\omega$  (what does this mean physically?) then the time-averaged power radiated across a section in  $x > x_0$  is the same as the time-average of (1) for this  $q(t)$  and a large factor  $4\pi c_0^2/\omega^2 A$  bigger than (5).