

Mixing, turbulence and chaos

– an introduction

E.J. Hinch

D.A.M.T.P., Cambridge University

May 8, 2003

1 Examples of mixing

1.1 Easy or not?

Mixing is essential for chemical reactions, in studying pollution and other processes in the atmosphere and ocean, and in very many industries. Our everyday experience of dissolving sugar in a coffee cup, which takes less than half a minute of stirring, would mislead us into thinking that mixing is easy, not worth the attention of a summer school.

First, however, we should note that the stirring is necessary: molecular diffusion would take over a day to diffuse sugar the centimeter length scale of the cup. To diffuse salt the 3 km depth of the oceans, molecular diffusion would take 10^8 yr, dangerously near the age of the Earth and much longer than the 10^4 yr needed with the stirring available.

Second, another diner-table experiment of trying to mix oil and vinegar reminds us of problems with immiscible fluids. Many foods, particularly the modern ‘lite’ products, are emulsions of fat in water. Some of the new specialists synthetic fibres are blends of incompatible polymers. For such immiscible fluids, the maximum permitted size of drops a (e.g. $1\ \mu\text{m}$ to ensure no separation of a colloidal under gravity) sets how vigorous the stirring must be through a minimum strain-rate $E \geq \chi/a\mu$, where χ is the interfacial tension and μ the viscosity of the continuous phase.

Third, vertical variations of density in a stratified fluid can significantly change mixing. Stirred at one level, a blob of fluid can only rise a certain height into less dense fluid before its kinetic energy is exchanged for potential energy. Thus stratification can inhibit mixing. On the other hand, if the density changes through changes in salt concentration as well as temperature, there is a ‘double-diffusive’ instability which causes strong mixing. This is thought to be important where the hot salty Mediterranean Sea flows out over the cold fresh Atlantic waters, and similarly for Norwegian fjords. There is also the possibility of internal gravity waves propagating in a stratified fluid. Where these waves break, they cause highly localised mixing, as aircraft discover in bumpy patches of ‘clear air turbulence’ miles away from any cloud or other obvious provocation.

Fourth, compared with the ease of stirring sugar in coffee, stirring very viscous liquids is less effective and requires much more energy. Practical examples include glass manufacture, molten polymers, heavy crude oils, industrial processing of food and domestic cooking, and minerals moving in the convection in the Earth’s mantle. There are also

problems of mixing powders in the pharmaceutical and food industries, and granular materials in civil engineering.

Fifth as we shall see later, mixing by stirring is considerable more difficult for substances with low values of molecular diffusivity. It is appropriate to compare the diffusivity of a substance (temperature, chemical species, Brownian particle) with the diffusivity of momentum, i.e. the kinematic viscosity ν . In water, momentum diffuses at $\nu = 10^{-6}$, temperature at $D_\theta = 10^{-7}$, salt at $D_{NaCl} = 10^{-9}$, and a $1\ \mu\text{m}$ colloidal particle (e.g. cream globules in milk) $D_{1\mu\text{m}} = 10^{-13}$. The SI units here are m^2s^{-1} . In air, momentum diffuses at $\nu = 10^{-5}$, temperature at $D_\theta = 10^{-5}$, oxygen at $D_{O_2} = 10^{-5}$, and a $1\ \mu\text{m}$ colloidal particle (e.g. smoke) $D_{1\mu\text{m}} = 10^{-11}$. We note that substances can diffuse many orders of magnitude more slowly than momentum. Thus smoke in a smoke-ring gives a false indication of the size of the vortex core.

1.2 Passive *vs* active mixing

The majority of recent studies of mixing, reflected in the contributions in this book, have been concerned with passive scalars which have no back-effect on the flow which is mixing them. This is appropriate in many applications. There are however circumstances in which the substance being mixed significantly changes the flow field which is mixing it.

In the oceans and atmosphere, the fluid density depends on temperature, and on salt in the oceans and on humidity in the atmosphere. A change in the density structure can lead to convection and a dramatically different flow. In the dynamics of the large scale motion of the atmosphere, there is a quantity called ‘potential vorticity’ which is advected and so stirred by the flow, and yet this variable determines the flow itself.

The phenomena of avalanches down mountains and of turbidity currents in the oceans involve flow resuspending and mixing particles to form a dense fluid which flows down hill, creating the mixing itself.

Chemical reactions can release heat which generates pressures sufficient to drive a flow. Thus explosions can occur in coal mines and in flour mills in which the flow mixes coal or flour dust from the floor up into the air where it burns explosively, creating the flow itself. A patch of turbulence can deform a slowly moving laminar flame front (typical velocity $1\ \text{m s}^{-1}$) so that the rate of release of energy per unit project area is sufficient to drive a compression wave with an amplitude which raises the temperature above the ignition temperature, triggering a rapid explosion wave (typical velocity $3\ \text{km s}^{-1}$).

Other examples where the substance being mixed actively changes the mixing flow are:– polymeric liquids resisting stretching by the flow, immiscible drops setting the small scale in the flow, cohesion of powders, and magnetic fields in recent models of the Earth’s dynamo.

1.3 Eulerian *vs* Lagrangian chaos

The nature of stirring divides into two classes depending on whether the Reynolds number of the flow is large or small. The Reynolds number is defined as the ratio of inertial forces to viscous forces in the flow, $Re = UL/\nu$, where U is the velocity scale, L is the length scale and ν is the kinematic viscosity.

At high Reynolds number, typically $Re > 10^3$, flows become turbulent. The small value of friction means that the cost of stirring is low, and also that the flow continues

sometime after stirring has stopped. Turbulence is characterised by having a wide range of length scales in the flow, and this ensures very efficient mixing.

At low Reynolds number, typically $R < 1$, friction is large, and so the cost of stirring is high. The flow stops immediately the mechanical agitation is switched off. There are only large scales in the flow, on the same length scale of the stirring device. Mixing can be very inefficient. The worst extremes are avoided if the fluid pathlines are chaotic. This so called ‘Lagrangian chaos’ or ‘chaotic advection’ can occur in regular smooth flows. To emphasise this difference, turbulence is sometimes called ‘Eulerian chaos’.

1.4 The need to mix

Good mixing is at the heart of many industrial processes, and the quality of the final product can depend entirely on the success of mixing. To make glass, tons of sand are mixed with tons of lime and some other mineral, first as dry powders of 1 mm size, and then in the molten phase in very viscous thermal convection. Any heterogeneity which survives this convection will produce optical imperfections and produce mechanical defects which disrupt the spinning of fibre-glass. The energy cost of continuing the thermal convection a little longer is measured in tens of MWs.

Mixing can be used to reduce concentrations of dangerous pollutants in the atmosphere, in estuaries and in ground water. Thus one can ask how high must a chimney be to avoid complaints from nearby housing, or where can one dump untreated sewage, or how can water quality in an aquifer be restored.

Mixing is essential for all chemical reactions. As well as the obvious chemical industry, one must remember fermentation and combustion are chemical reactions, if rather complex.

The surface temperature of the of the sea depends on the effectiveness of vertical mixing after a large storm or other large disturbances. Anomalies in the surface temperature, which persist for days, can lead to significant modifications of weather patterns, in so called ‘blocking events’.

Finally, the existence of the ozone hole, localised to the winter pole of the stratosphere, depends on mixing processes in the upper atmosphere.

1.5 Engineering

Many recent publications on mixing seem to me to have lost sight of why the subject is being studied. An understanding of how mixing takes place must produce a simple usable quantification of the process. Of course the quantities of interest will vary a little from application to application, but three themes are always present.

First, one needs to know how efficient the mixing is. In some applications, it may be required that the mixing is total, producing a uniform concentration throughout the container, for example in the pharmaceutical industry. In other applications, it may be satisfactory to produce a uniform mixing within only part of the container, e.g. to ensure a guaranteed quantity of chemical production. In pollution control or to avert an explosion, it may be necessary just to reduce the maximum concentration below a critical level. A simple measure of the degree of mixing is to compare the r.m.s. fluctuations of concentration with the mean level, $\overline{c'^2}/\bar{c}^2$.

Second, it is important to know how long it takes to achieve a specified degree of mixing. This time for mixing has to be compared with the time for chemical reactions.

Similarly, continuously operated stirred-tank reactors must be designed so that the residence time exceeds the mixing time, even exceeding it several fold over when a high degree of mixing is required. (One needs here to consider the probability distribution of residence times.) A crude overall measure of the time of mixing is provided by the rate of decrease of the r.m.s. fluctuations in concentration, $\overline{c'^2} \left[d(\overline{c'^2})/dt \right]^{-1}$.

Finally, but most important, one needs to count the cost of the mixing operation. The energy consumption in mixing at high Reynolds numbers is usually not a significant part of the total manufacturing costs. Mixing at low Reynolds numbers is however expensive, and careful design is necessary to avoid ineffectual dissipative flow. Cost may have to be reckoned per unit volume of mixture or per unit time.

1.6 Further examples

There is no general or universal theory of mixing, because applications are limitless and each brings its own peculiar twist.

Gas fluidised beds suffer from bubbling, in which packets of gas rise without contacting the catalytic particles. These undesirable but unavoidable bubbles do however play a helpful role in stirring the particles, which enables one slowly to rejuvenate them as they visit a small region.

Yield fluids, such as pastes and many semi-solid foods, have the unfortunate behaviour of flowing only in a finite region nearby the paddles. Thus it is necessary to move a kitchen whisk around a cooking bowl to ensure that all the material comes near to the mixer blades.

While most of this book is devoted to mixing within a volume, one must not forget the classical studies and results for heat and mass transfer across boundaries, including in particular Levich's $Pe^{-1/3}$ boundary layer when the diffusivity is smaller than the kinematic viscosity. One interesting geophysical problem is the heat and humidity transfer at the sea surface, which is greatly enhanced by the presence of bubbles (more than doubling the surface area of the oceans). A novel design for a heat exchanger, patented by M.R. Mackley, uses eddies shed in pulsed flow through baffles in a pipe to scour and replenish the boundary fluid – a good application of Lagrangian chaos.

A topic explored in depth at the earlier Summer School on 'Disorder and Mixing' is dispersion by the random flow in a porous medium. This mixing can usually be characterised by a diffusivity $D \sim \bar{u}a$, where \bar{u} is the volume flow rate per unit surface area and a is the grain size of the medium. There is a similar phenomenon of Taylor dispersion of a passive scalar in laminar pipe flow with a diffusivity $D = D_{mol} + 0.01U^2a^2/D_{mol}$, where D_{mol} is the molecular diffusivity, U is the average velocity and a is the radius of the pipe. This Taylor dispersion can be applied to mixing fresh air in the lungs.

Finally in a laminar shear flow of a suspension, particles move across streamlines due to random collisions. This motion is described by a diffusivity $D = 0.3\gamma a^2$ if $0.1 < \phi < 0.5$, where γ is the shear-rate, a is the size of the particles, and ϕ is the volume fraction of particles.

2 The issues

2.1 Molecular diffusion

Consider the molecular diffusion of a passive scalar. Let $c(x, t)$ be the concentration of the substance and D its molecular diffusivity. The appropriate diffusion equation in one dimension is

$$c_t = Dc_{xx}.$$

Let the initial conditions be $c = 0$ in $x < 0$ and $c = 1$ in $x > 0$. The well known solution of the diffusion problem is

$$c(c, t) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{Dt}}\right),$$

with $\text{erf}(z)$ the standard error function.

From this solution, or from dimensional analysis of the governing equation, we identify the distance over which substance has diffused

$$\delta = \sqrt{Dt}.$$

The concentration has a value $c = 0.5 \pm 0.3$ at $x = \pm\delta(t)$ and a value $c = 0.5 \pm 0.45$ at $x = \pm 3\delta(t)$.

For the diffusion of heat in air, this diffusion distance is 0.14 mm after 1 ms, 4 mm in 1 s, 0.3 m in 1 h. For the diffusion of salt in water, $\delta = 1 \mu\text{m}$ in 1 ms, $30 \mu\text{m}$ in 1 s, 2 mm in 1 h, and 0.2 m in 1 day. From these examples, we learn that molecular diffusion is very fast at early times, as $1 \mu\text{m}$ is more than 1000 molecules in size, but very slow as time increases. This slowness is the heart of the mixing problem.

2.2 Value of the diffusivity D

For dilute gases, simple kinetic theory gives a prediction for the value of the molecular diffusivity

$$D = 0.3\bar{u}h.$$

Here \bar{u} is the mean thermal velocity of the molecule, which can be found from the Boltzmann temperature kT and the mass of the molecule m , $\frac{1}{2}m\bar{u}^2 = \frac{1}{2}kT$. The mean-free-path h is related to the radius of the molecule a and the density of the gas ρ , $4\pi a^2 h \rho = m$ for a pure gas.

For colloidal particles in Brownian motion, the Stokes-Einstein formula for diffusivity of particle is

$$D = \frac{kT}{6\pi\mu a},$$

where μ is the dynamic viscosity of the suspending fluid and a is the radius of the assumed spherical particles. There are corrections to the Stokes-Einstein formula if the volume fraction of the particles is not small.

For turbulent motion, one can define an eddy diffusivity by analogy with kinetic theory

$$D = u'L,$$

where u' is the r.m.s. value of the velocity fluctuations in the fluid and L is the length scale of eddies with this fluctuating velocity.

2.3 Dispersion of a random walk

Consider a random walk of uncorrelated steps of size Δx , each taking a time Δt . After a lapse time of t , the number of steps taken is $t/\Delta t$. The variance in the displacement from the starting position will be the sum of the variances of the uncorrelated steps, i.e. $\Delta x^2 \times t/\Delta t$. Note that the variance increases linearly in time (in this normal diffusion process). The coefficient of the linear growth is defined to be the diffusivity of the random walk

$$D = \frac{d}{dt} \left(\frac{1}{2} \overline{x^2(t)} \right).$$

The over-bar denotes an average over several experiments. A constant value for the D is seen only after several correlation times Δt .

Taylor's calculation of the eddy diffusivity in turbulence can now be presented. Proceeding formally,

$$\frac{d}{dt} \left(\frac{1}{2} \overline{x^2(t)} \right) = \overline{\dot{x}(t)\dot{x}(t)} = \int_0^t \overline{\dot{x}(t')\dot{x}(t)} dt'$$

The diffusivity attains its constant value only after several correlation times. (When the integral fails to converge, due to slowly decaying correlations, the diffusion becomes 'anomalous' with $\overline{x^2(t)} \sim kt^\mu$ with $\mu \neq 1$.) Thus

$$D = \overline{u'^2} T,$$

with $\overline{u'^2}$ the mean square of the velocity fluctuations and T the integral-correlation time

$$T = \int_0^\infty \overline{\dot{x}(t)\dot{x}(0)} dt / \overline{u'^2}.$$

Note that the velocity correlation needs to be computed as seen by a particle moving with the fluid, and hence T is known as the Lagrangian integral-correlation time. The length scale L , introduced at the end of the previous subsection as the size of the eddies, can now be defined as $L = u'T$, i.e. the distance one would move at the r.m.s. velocity u' during the correlation time T .

Taylor's calculation can also be used to derive the Stokes-Einstein diffusivity for a colloidal particle. The thermal velocity fluctuations of the particle are given by $\frac{1}{2}m\overline{u'^2} = \frac{1}{2}kT$. The velocity remains correlated while friction forces dissipate inertia according to

$$m\ddot{x} + 6\pi\mu a\dot{x} = 0,$$

i.e.

$$\overline{\dot{x}(t)\dot{x}(0)} = \frac{kT}{m} \exp(-6\pi\mu at/m).$$

Evaluating the integral, one recovers the Stokes-Einstein formula given in the last section.

A further application of Taylor's calculation gives the diffusivity of Taylor dispersion described in §1.6. In pipe flow, the variation of the velocity between the walls and the centre of the pipe gives $u' = \bar{u}$. The velocity of a passive scalar remains correlated until it has had time to diffuse molecularly across the cross-section of the pipe, i.e. $T = a^2/D_{mol}$. Hence we estimate $D = O(\bar{u}^2 a^2/D_{mol})$. The numerical coefficient of about 10^{-2} comes from a more careful analysis.

2.4 The problem with molecular diffusion

The problem with purely molecular diffusion is that it is far too slow, as seen in §2.1, the volume of fluid mixed at a molecular level is

$$V = A\sqrt{Dt},$$

where A is the interfacial area between the species being mixed. With the diffusion distance \sqrt{Dt} increasing slowly in time, it is clear that the only way to improve mixing is to increase the interfacial area. Large factors such as 10^6 have to be contemplated.

One useful measure of the effect of stirring is the so called ‘striation thickness’

$$V/A,$$

which can be thought of as the thickness of, or between, sheets of unmixed fluid. The aim is to reduce this thickness to distances over which molecular diffusion can act in the time available, i.e. to \sqrt{Dt} .

2.5 Benefit and limitation of shear

Consider a dyed region of fluid, initially inside a sphere of radius a , deforming in a simple shear flow $\mathbf{u} = (\gamma y, 0, 0)$. After time t , the sphere becomes an ellipsoid

$$(x - \gamma ty)^2 + y^2 + z^2 \leq a^2,$$

with semi-diameters $a\left(\sqrt{1 + \gamma^2 t^2/4} \pm \gamma t/2\right)$ and a . The surface area of the ellipsoid is a complicated elliptic function, which for large shears $\gamma t > 3$ becomes $2\pi a^2 \gamma t$ (within 10%). Thus shear increases the interfacial area between the dyed fluid inside and the clear fluid outside the initial sphere, which is beneficial to mixing. Another way to look at this is to note that the thickness of the dyed fluid, i.e. the striation thickness, is reduced, at large shears $\gamma t > 3$ becoming $2a/\gamma t$.

There are two limitations to the beneficial effect on mixing of a simple shear flow. First, the growth in interfacial area (or equivalently the decrease in the striation thickness) is only linear in time. This to achieve an increase by a factor 10^6 , a time of $10^6/\gamma$ must be used. This is unreasonably long.

Second, the simple shear $\mathbf{u} = (\gamma y, 0, 0)$ is a local idealisation, and does not continue indefinitely. After many shear times, one must consider the wider global flow and this can limit mixing. For example consider a sphere of dyed fluid of radius a released in a Couette device at a radial position r_0 and axial position z_0 . After a large number $O(r_0/a)$ of turns the dye will be completely spread around the torus $(r - r_0)^2 + (z - z_0)^2 < a^2$ by the shearing flow. To mix to other radial or axial positions, however, only molecular diffusion can be used.

2.6 Stretching of lines, areas and volumes by flow

We consider the evolution of small lines, areas and volumes in a flow $\mathbf{u}(\mathbf{x}, t)$. It is the difference in velocity across these elements which causes change. Hence we need to consider the velocity gradient $\nabla\mathbf{u}$. This is split into its symmetric and antisymmetric parts, the strain-rate \mathbf{E} and vorticity $\mathbf{\Omega}$,

$$\nabla\mathbf{u} = \mathbf{E} + \mathbf{\Omega}, \quad \text{with} \quad \mathbf{E}^T = \mathbf{E} \quad \text{and} \quad \mathbf{\Omega}^T = -\mathbf{\Omega}.$$

Now a short line of fluid from \mathbf{x} to $\mathbf{x} + \delta\mathbf{l}$ will move in δt to the line from $\mathbf{x} + \mathbf{u}(\mathbf{x})\delta t$ to $\mathbf{x} + \delta\mathbf{l} + \mathbf{u}(\mathbf{x} + \delta\mathbf{l})\delta t$. Thus the line element $\delta\mathbf{l}$ has become $\delta\mathbf{l} + \delta\mathbf{l} \cdot \nabla\mathbf{u}\delta t$, i.e.

$$\dot{\delta\mathbf{l}} = \delta\mathbf{l} \cdot \nabla\mathbf{u}.$$

The component of the change in the direction of $\delta\mathbf{l}$ is the rate of stretching of the line, and this is entirely due to the straining motion \mathbf{E} . The component of the change perpendicular to $\delta\mathbf{l}$ represents a rotation of the line and comes from both the straining \mathbf{E} and the vorticity $\boldsymbol{\Omega}$. To examine further the stretching, we write $\delta\mathbf{l}$ as a length δl and a direction \mathbf{n} of a unit vector, $\delta\mathbf{l} = \mathbf{n}\delta l$. One then obtains the fractional change of length

$$\frac{d}{dt} \ln \delta l = \frac{\dot{\delta l}}{\delta l} = \mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}.$$

It should be noted the \mathbf{n} rotates in the flow, and so the rate of stretching of the line may not be constant even in a steady flow.

Volumes of fluid remain constant in time in incompressible flows. This fact can be used to find how area elements $\delta\mathbf{A}$ change in flow. If $\delta\mathbf{l}$ is a line element, the volume produced by displacing area $\delta\mathbf{A}$ through the distance $\delta\mathbf{l}$, i.e. $\delta\mathbf{A} \cdot \delta\mathbf{l}$, remains constant in the flow. Thus

$$0 = \frac{d}{dt} (\delta\mathbf{A} \cdot \delta\mathbf{l}) = \dot{\delta\mathbf{A}} \cdot \delta\mathbf{l} + \delta\mathbf{A} \cdot \dot{\delta\mathbf{l}}.$$

Using the earlier result for $\dot{\delta\mathbf{l}}$, we obtain

$$\dot{\delta\mathbf{A}} = -\nabla\mathbf{u} \cdot \delta\mathbf{A}.$$

To examine the important stretching of the area, we write $\delta\mathbf{A} = \mathbf{n}\delta A$ with \mathbf{n} the unit normal perpendicular to the surface. The fractional change of the area is then

$$\frac{d}{dt} \ln \delta A = \frac{\dot{\delta A}}{\delta A} = -\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}.$$

We note that if at one instant a line element $\delta\mathbf{l}$ is perpendicular to an area $\delta\mathbf{A}$, then the fractional increase in the length of the line would be equal to the fraction *decrease* in the size of the area. This is however at one instant: the line and area will rotate differently and will not remain perpendicular.

In two dimensions, the stretching of lines and areas are equivalent. In three dimensions, the fractional rate of stretching lines tends on average to be greater than the fractional rate of stretching areas, because, while one principal direction in the area will be stretched as fast as the line, the orthogonal direction will not stretch as fast and may even contract.

2.7 Efficiency of stretching

Because molecular diffusion alone is slow to mix, a stirring flow is required to increase greatly the interfacial area. An appropriate measure of the rate of increase in area is the Liapounov exponent, defined for a small area moving with the flow

$$\sigma_A = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\delta A(t)}{\delta A(0)}.$$

In principle this exponent depends on the initial position and the initial orientation of the small area. There is also an assumption that the initial area $\delta A(0)$ is sufficiently small for the area to remain small compared with the scale of the flow at later times. With a non-zero Liapounov exponent, the area increases exponentially in time $\delta A(t) \sim \delta A(0) \exp(\sigma_A t)$. Roughly, the exponent is the long-time average of the component of the strain-rate $\sigma_A = -\langle \mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n} \rangle$, where \mathbf{n} rotates with the flow and \mathbf{E} varies as the area moves through the flow.

In steady simple shear, area increases linearly in time, and so $\sigma_A = 0$. In steady axisymmetric extension, however the area rotates to an orientation which benefits fully from the stretching, $\sigma_A = E$.

One measure of the *efficiency* of a stirring apparatus is to compare the average Liapounov exponent for the increase in area with the average strain-rate

$$\langle \sigma_A \rangle / \langle E \rangle.$$

The exponent would be averaged over different initial positions and orientations, while the strain-rate would be averaged over the flow. In very viscous liquids, it is undesirable to have any straining motion producing viscous dissipation while not also producing stretching. Thus a Couette device is a very inefficient mixer (zero efficiency). Well designed stirrers can achieve 30% efficiencies.

3 Turbulent mixing

3.1 Examples

To keep this brief introduction firmly anchored in reality, I would like to start by referring to the excellent collection of pictures, *An Album of Fluid Motion*, collected and published by M. Van Dyke (1982), hereafter referred to as MVD. We will look at four practical examples of turbulent mixing flows – the mixing layer, a jet, flow behind a grid, and pipe flow.

Pictures MVD 176 and 177 (by Konrad 1976 and by Rebollo 1976) show a turbulent mixing layer where two separate uniform streams join. At high Reynolds numbers $O(10^6)$, one sees a mixing zone between the two streams, with a width growing linearly with distance downstream from the splitter plate, the point where the two stream first join. Spanning the width of the layer are large scale eddies, similar in general form to the nonlinear Kelvin-Helmholtz instability. These large scale eddies change little with Reynolds number. Superimposed on the large eddies are small scale eddies, which become finer as the Reynolds number increases. Picture MVD 177 shows the length scale of the small eddies $\frac{1}{50}$ th that of the width of the mixing layer.

Picture MVD 166 (by Dimotakis, Lye & Papantoniou 1981) shows a jet of dyed water immersing into a bath of water at a Reynolds number of 2300. The width of the jet of mixed water grows linearly with distance downstream from the nozzle. Superimposed on large scale eddies which span the width of the jet are smaller scale eddies of several scales, the finer ones having a length scale about $\frac{1}{50}$ th the width of the jet.

Pictures MVD 152 and 153 (by Corke & Nagib) show the turbulence behind a grid, which can be dragged through a mixture to enhance the rate of mixing. At a Reynolds number of 1500, one sees near to the grid large eddies on the scale of the grid. A little behind, the flow becomes dominated by much smaller eddies. The length scale of these random fine eddies appears to increase at some distance downstream of the grid.

Van Dyke's book does not give any pictures of turbulent pipe flow, because of practical difficulties in making the images. Equivalent however are the pictures MVD 157, 158, 162 and 163 (by Corke, Guezennec & Nagib; Falco 1977; Falco; Head & Bandyapadhyay 1981) of a turbulent boundary layer on a flat plate. Here one sees fine scale eddies, possibly originating at the wall. There are no apparent large scale eddies, except perhaps for some large scale organisation of regions with and without small scales in figure MVD 157. This intermittent nature of some turbulence can be important. In pipe flow one can detect coordinated activity across the width of the pipe, i.e. large scale eddies spanning the diameter.

3.2 What is turbulence and why?

There is no generally accepted definition of turbulence and no explanation of why it exists. It is clear from the pictures referred to above that turbulent flows are random (or chaotic) in space and possess a wide range of length scales superimposed on one another. The flows are equally random in time, with a wide range of time scales superimposed. Thus the complex flow structures cannot be thought of as simply propagating. Instead each eddy seems first to evolve and then to decay during the time for fluid to go once around the eddy, the eddy turnover time. Some of the large structures may persist several turnover times. The wide range of length and time scales, which characterise turbulence and which distinguish it from the 'Lagrangian turbulence' or 'chaotic advection' in section 4, contribute to the efficiency of mixing by turbulent flows.

Why are virtually all high-Reynolds-number flows turbulent is a very good and simple question with no widely accepted, generally applicable answer. Certainly one can observe that the advection term $\mathbf{u} \cdot \nabla \mathbf{u}$, which dominates the smoothing viscous term at high Reynolds numbers, has the tendency to enhance spatial gradients, i.e. produce small scales. It is often speculated that most/all incompressible Euler flows (totally ignoring the viscous term) will produce a shock with a finite velocity discontinuity (or more plausibly a discontinuity in velocity gradient) in a finite time, although no convincing example has yet been produced.

At large Reynolds numbers, sharp shear layers are seen on the edge of the large scale eddies. These shear layers are vigorously unstable to small scale disturbances. There is the Kelvin-Helmholtz instability for fairly flat shear layers, see pictures MVD 81 and 55 (by Pierce 1961 and Werlé 1980), and for curved shear layers the Taylor-Görtler and Taylor-Couette instabilities, see pictures MVD 144 and 128 (by Bippes 1972 and Koschmieder 1979). The length scale of these fast growing instabilities is that of the thickness of the shear layer, while the velocity difference across the layer gives the velocity scale. One can envisage the eddies of these instabilities suffer themselves from secondary finer instabilities, and so on until viscosity can no longer be ignored. The instability mechanism at the largest scales of the flow may be particular to the special nature of the apparatus which produces the large scale flow. The mechanisms at smaller scales may be more universal, appearing in most flows.

3.3 Macro and micro scalings

The scales of the large eddies are set by the geometry and speed of the stirring mechanism, while the cut-off scales of the small eddies are determined by the action of viscosity. We discuss the details of the different large eddies in sections 3.6–3.9 for the different mixing

flows introduced in section 3.1. Here we concentrate on the small scales for a flow with large eddies of given velocity, length and time scales, $U_L, l_L, T_L = l_L/U_L$. There is an assumption due to Kolmogorov, which is probably only partially true, that the small eddies are universal for given U_L, l_L and viscosity ν .

The important Kolmogorov microscale for the smallest eddies is based on a further assumption that the smallest eddies depend only on the rate at which energy is put into the large eddies, i.e. on one particular combination of U_L and l_L . An argument can be made that friction only acts on the smallest scale, that the sole rôle of the smallest scale is to dissipate all the energy, that the rate of dissipation must equal the supply of energy, that energy is supplied only at the large scale. The rate of dissipation is measured per unit mass, ϵ , and can be related to the macroscales by assuming that a significant fraction of the kinetic energy in the large eddies is dissipated in the turnover time of the large eddies, i.e. per unit time

$$\rho\epsilon = \rho U_L^2/T_L, \quad \text{i.e.} \quad \epsilon = U_L^3/l_L.$$

The dimensions of this dissipation per unit mass ϵ are L^2T^{-3} , while the dimensions of the kinematic viscosity ν are L^2T^{-1} . Hence by dimensional analysis, we obtain the velocity, length, time and strain-rate scalings of the Kolmogorov microscale

$$U_K = (\nu\epsilon)^{1/4}, \quad l_K = (\nu^3/\epsilon)^{1/4}, \quad T_K = (\nu/\epsilon)^{1/2} \quad \text{and} \quad E_K = (\epsilon/\nu)^{1/2}.$$

Introducing the Reynolds number of the large scale eddies $Re = U_L l_L/\nu$, we have

$$U_K = U_L Re^{-1/4}, \quad l_K = l_L Re^{-3/4}, \quad T_K = T_L Re^{-1/2}, \quad \text{and} \quad E_K = E_L Re^{1/2}.$$

A common notation for the Kolmogorov microscales are η for l_K , τ for T_K and v for U_K .

In practice the Kolmogorov micro length scale can be quite small. It is however bigger than the mean-free-path by a factor $Re^{1/4}/Ma$ where Ma is the Mach number. Here we have used the expression for the mean-free-path $h = \nu/\bar{u}$, and set the mean thermal speed equal to the speed of sound.

The Kolmogorov microscale for the smallest eddies depends on the velocity and length scales of the large eddies only in the combination $\epsilon = U_L^3/l_L$. A second microscale, the Taylor microscale uses a different combination to yield a slightly larger scale. The Taylor microscale can be thought of as the boundary layer thickness on the edge of a large eddy, i.e. $l_T = \sqrt{\nu t}$ with $t = T_L = l_L/U_L$ the turnover time of the large eddy. Hence

$$l_T = (\nu l_L/U_L)^{1/2} = l_L Re^{-1/2}$$

using the Reynolds number of the large eddies. A common notation for the Taylor microscale is λ for l_T .

So far we have considered the largest and the smallest eddies, and these two extremes are sufficient to discuss the process of turbulent mixing. An additional hypothesis of Kolmogorov is that eddies exist at all scales, and that these intermediate scales are too large to depend on viscosity and too small to depend on the details of the large eddies. Thus it is assumed that they only depend on the energy being transferred from the large to the small scales, ϵ , i.e. the energy to be dissipated ‘cascades through each intermediate scale’. Dimensional analysis then gives that the velocity scale of intermediate eddies of length scale l is $(\epsilon l)^{1/3}$ and the time scale is $(l^2/\epsilon)^{1/3}$. Much more can and will be said about the spectrum of eddies, although it is not clear to me that independent dynamical entities exist at all scales.

3.4 Mixing with the Taylor eddy diffusivity

We now turn to mixing by turbulent flows. Now the turnover time of the smallest eddies T_K is shorter than the turnover time of the large eddies T_L by the factor $Re^{-1/2}$. Hence mixing takes place faster and more efficiently on small scales than on large scales. (As will be explained in the next section, the proceeding remark is a little misleading when the molecular diffusivity D is very small.) The process limiting the rate of mixing is therefore mixing the fluid from its initial separated state to become homogeneous on the scale of the large eddies.

Large scale mixing is described by the the Taylor eddy diffusivity $D = U_L l_L$, see section 2.3. For a container of height H , the time for eddy diffusion is then

$$t_{\text{macro}} = H^2/D = t_L H^2/l_L^2.$$

This is many times, H^2/l_L^2 times, the turnover time of the large eddies when the apparatus is larger than the large eddies, as might occur with small paddles or a small mixing grid. If the source of the turbulence is small, one also has to worry whether the turbulence intensity, and hence the eddy diffusivity, is uniform throughout the container.

3.5 The problem of low molecular diffusivity D

As discussed at the end of section 1.1, the value of the molecular diffusivity D can be very much smaller than the diffusivity of momentum ν , by 6 orders of magnitude in the case of the diffusion of colloidal particles. When the diffusivity is small, turbulent mixing will homogenise the fluid down to the length scale of the large eddies l_L in a time t_{macro} , as described in the previous section. Because the small eddies turnover much faster, the fluid will be homogenised down to the scale of the small eddies l_K after a negligible time delay. Mixing down to this scale is however considerably short of mixing down to the molecular scale, or the particle scale in the case of colloidal particles.

The time for purely molecular diffusion to homogenise the fluid within the smallest eddy is $l_K^2/D = t_K \nu/D$, i.e. the small eddy turnover time multiplied by the Schmidt number $Sc = \nu/D$. Fortunately there is a faster mechanism for mixing within the smallest eddy. The rate of straining within the smallest eddy is e_K and this straining increases the interfacial area between unmixed regions exponentially like $e^{\langle\sigma_A\rangle t}$, where $\langle\sigma_A\rangle \approx 0.3e_K$ is the Liapounov exponent of section 2.7 for area increase. The ‘striation thickness’ between unmixed regions within a small eddy therefore decreases like $l_K e^{-\langle\sigma_A\rangle t}$, and will equal the molecular diffusion distance \sqrt{Dt} at a time

$$t_{\text{micro}} = 3t_K \ln(\nu/D)^{1/2}.$$

By this time the interfacial area will have increased by a factor $(\nu/D)^{1/2}$, i.e. the striation thickness will have reduced from the size of the smallest eddies to $\eta_\theta = l_K(D/\nu)^{1/2}$.

Comparing the time t_{macro} to homogenise the container down to the length scale of the large eddies with the above time t_{micro} to homogenise down to the molecular diffusion scale within the smallest eddies, we see that the low value of the molecular diffusivity is a problem limiting the mixing process if

$$Re^{-1/2} \ln(\nu/D)^{1/2} > H^2/l_L^2.$$

3.6 The turbulent mixing layer

To complete this rapid survey of turbulent mixing, it remains to give estimates for the velocity and length scales of the large eddies for the four typical turbulent flows described in section 3.1.

For the mixing layer between two streams of fluid travelling parallel at speeds U and $U + \Delta U$, the typical velocity of the large eddies is about a quarter of the slip velocity, i.e.

$$U_L = \frac{1}{4}\Delta U.$$

Picture MVD 177 shows the mixing zone increasing linearly with distance downstream, keeping within a wedge of angle about 20° . Thus the length scale of the large eddies is about

$$l_L = \frac{1}{17}x,$$

where x is the distance downstream of the splitter plate.

3.7 The turbulent jet

We consider just the case of a three-dimensional jet with volume flux Q issuing from a circular pipe. Thus at the nozzle the jet velocity is $U_0 = Q/\pi a^2$.

Picture MVD 166 shows that the turbulent jet spreads within a cone of about 20° in a fashion similar to the mixing layer. The large scale eddies span much of the width of the jet and so again have a length scale

$$l_L = \frac{1}{15}x,$$

where x is the distance downstream of the nozzle.

The typical velocity in the large eddies is about half the mean velocity of the jet on its axis. The latter decreases downstream as the initial momentum flux $\rho U_0^2 a^2$ is spread over an increasing volume of fluid within the 20° cone. At high Reynolds numbers, the momentum flux must be the same at each downstream section of a steady jet, and so the velocity of the mean flow must decrease inversely proportionally with distance downstream. Thus the velocity of the large eddies is

$$U_L = 13U_0 a/x.$$

The entrainment of fresh fluid into the turbulent jet as it spreads within the 20° cone is of course a useful mixing which is employed to dilute many times over concentrated dye. The concentration within the spreading jet reduces as $c_0 a/x$, in order that the flux is the same at each downstream section.

Thus one can ask where on the ground downstream of a chimney is the concentration of pollution greatest, assuming that the flow from the chimney can be modelled as a turbulent jet (appropriate only under certain conditions of wind and stratification). The ground concentration will be highest where the spreading cone first hits the ground, i.e. at a distance downstream approximately five times the height h of the chimney. Further the ground concentration is roughly $c_0 a/15h$, so one can decide how high the chimney must be to avoid a dangerous level.

3.8 Turbulent wake behind a cylinder

Cylinders occur in mixers most often in the form of a grid which is dragged through the fluid. Here we consider just an isolated cylinder, although the scalings can be equally applied to the turbulence behind a grid.

While one might at first think that the spreading of the momentum deficit in the wake behind a cylinder ought to be very similar to the spreading of the momentum flux in a jet, there are in fact considerable differences: the velocity scale of the large eddies in the wake increases rather than decreases downstream, and the length scale of the large eddies increases less rapidly in the wake compared with the jet. The difference in the two flows is clearly apparent in the frame in which the fluid at infinity is at rest. The jet is stationary on average, with spreading resulting from the flow downstream within the jet. The wake on the other hand is transient, with fluid being displaced only a short distance as the cylinder passes. The spreading of the wake is due to lateral diffusion in time rather than displacement downstream.

We find the length l_L and velocity U_L scales of the large eddies by combining two arguments. First, a geometrical argument that the velocity fluctuations result from deflecting the free-stream flow U_0 through an angle x/l_L , where x is the distance downstream from the cylinder, i.e. $U_L = U_0 x/l_L$. Second, the deficit in the momentum flux in the wake $\int \rho U_0 U_L dy$ per unit length of the cylinder must equal the drag on the cylinder $\frac{1}{2}\rho U_0^2 a$, i.e. $\rho U_0 U_L l_L = \rho U_0^2 a$. Solving we find

$$U_L = 1.6U_0\sqrt{\frac{a}{x}} \quad \text{and} \quad l_L = 0.25a\sqrt{\frac{x}{a}}.$$

A consequence of these estimates is that the Reynolds number based on the large eddy scales $U_L l_L/\nu = 0.4U_0 a/\nu$ does not change downstream in the wake, a so called self-similar property of this flow. A second consequence is that the eddy diffusivity is also constant $D = U_L l_L = U_0 a$, and so the sideways diffusion $l_y = \sqrt{Dt}$ after the time $t = x/U_0$ to flow a distance x downstream is $l_y = \sqrt{ax}$, i.e. the size of the large eddies.

3.9 Turbulent pipe flow

Turbulence in a pipe, and in other confined geometries such as the turbulent boundary layer on a rigid plate, is considerably different from the turbulence in the open geometries of the previous three sections. The origin of the turbulence in the pipe is the friction of the wall, i.e. the no-slip boundary condition exerted by viscosity. Thus energy is feed into the turbulence at the small scale, at the scale at which viscosity is important, rather than at the large scale, as in the previous three geometries. One consequence of this difference in the driving of the turbulence is that the scale of the velocity fluctuations is the same for eddies of all scales. Traditionally this magnitude of velocity fluctuations is given the symbol u_* , and is called the ‘friction velocity’.

The friction velocity is related to the mean pressure gradient dp/dz in the pipe by the following argument. Consider a pipe of length L and radius a . The difference in the pressure force acting on the two ends, $\pi a^2 L dp/dz$, must balance the tangential friction stress τ exerted on the wall $2\pi a L \tau$. This friction stress is expresses as a Reynolds stress $\tau = \rho u_*^2$ using the friction velocity. Hence

$$\rho u_*^2 = a \frac{dp}{dz}.$$

The length scale of the smallest eddies l_* depends on whether the pipe has smooth walls or rough. If it is hydrodynamically smooth, as in careful laboratory experiments, then the size of the smallest eddies is such that viscous and inertial forces are comparable for the velocity fluctuations

$$l_* = \nu/u_*.$$

If the pipe has a roughness with an amplitude greater than this value, which is common in practical applications, then l_* is replaced by the height of the roughness elements.

The size of the largest eddies depends on the radial position in the pipe. In MVD 157 & 163 one can see that it is roughly half the distance from the wall, i.e.

$$l_L = 0.4(a - r).$$

Finally it is necessary to relate the frictional losses represented by u_* to the volume flow through the pipe, i.e. to the mean flow \bar{u} . Now the mean tangential stress, which is exerted by molecular diffusion and eddy transport of momentum, varies linearly across the pipe from its wall value τ (by using the argument for u_* applied to a cylinder of arbitrary radius within the pipe). The eddy diffusivity of momentum by the large eddies is given by section 2.3 as $0.4u_*(a - r)$. Thus

$$\rho(\nu + 0.4u_*(a - r))\frac{d\bar{u}}{dr} = \rho u_*^2 \frac{r}{a}.$$

Integrating, and adjusting the constants to fit experimental observations, we obtain the ‘log law of the wall’

$$\bar{u} = u_* \left(2.5 \ln \frac{a}{l_*} + 6 \right).$$

An application of the above scalings is to estimate how far downstream dye must travel before it becomes well mixed in a turbulent pipe. Let the pipe be hydrodynamically smooth, so $l_* = \nu/u_*$, and take the Reynolds number to be $Re = \bar{u}a/\nu = 6000$. Solving the last equation in the previous paragraph, we find $u_* = \frac{1}{20}\bar{u}$. If the molecular diffusivity of the dye is not too small, so that we need only consider mixing on the large scales, then the time to diffuse across the radius a with eddy diffusivity u_*a is a/u_* . During this time the fluid has moved downstream at the mean velocity \bar{u} a distance $20a$.

4 Lagrangian mixing

4.1 The problem

Mixing in viscous liquids at low Reynolds numbers can be difficult. Specially designed mixers are required to compensate for the lack of the small length scales in the flow that are generated internally at large Reynolds numbers. The large value of friction means that energy consumption is high, and this places a premium on the efficiency of mixing. Industrial applications include blending polymers, homogenising glass and processing food.

Mixing at low Reynolds numbers becomes difficult when the molecular diffusivity of the substance to be mixed is so small that the Péclet number UL/D is large. As seen in section 2.2, the diffusivity of colloidal particles is inversely proportional to the viscosity, and so it is not uncommon in viscous liquids for the Reynolds number to be small while the Péclet number is large.

When the Péclet number is large, material is mainly advected by the flow and it diffuses very slowly. It is therefore essential that the flow field has the property of efficiently dispersing material throughout the volume. This requires that two blobs of fluid which start nearby one another should separate rapidly. The appropriate measure of the rate of this process is the Liapounov exponent $\langle \sigma_A \rangle$ introduced in section 2.7.

4.2 Chaotic pathlines

The ‘big idea’ of Lagrangian chaos is the observation that velocity fields $\mathbf{u}(\mathbf{x}, t)$ which are smooth, regular and generally boring when viewed in the traditional Eulerian fixed-laboratory frame can produce fluid pathlines $\mathbf{x}(\mathbf{x}_0, t)$ [the solution of $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t)$ in $t \geq 0$ with $\mathbf{x} = \mathbf{x}_0$ at $t = 0$] which uniformly fill a volume in an ergodic way. These chaotic fluid pathlines disperse material efficiently, i.e. give good mixing with nonzero Liapounov exponents.

Steady two-dimensional flows cannot have chaotic fluid pathlines: in the language of dynamical systems they are called ‘integrable’. Because in a steady flow the pathlines are streamlines, and because in two dimensions streamlines cannot cross (except at points where the velocity vanishes so that fluid blobs will never reach them), almost all fluid pathlines are closed. Thus dye released in one region will remain limited to the streamlines which pass through the initial dyed region. Mixing is therefore inadequate.

Chaotic fluid pathlines can however occur in time-periodic two-dimensional flows, in steady three-dimensional flows, and in more general flows. The special case of time-periodic two-dimensional flows has been much studied due to the availability of powerful methods of analysis.

A very good introduction to the subject of mixing by Lagrangian chaos is given by the textbook by J.M. Ottino *The kinematics of mixing: stretching, chaos and transport*, hereafter denoted by JMO. I will refer to several of the excellent collection of pictures in that book.

4.3 Changing the direction of shear

We saw in section 2.5 that a steady simple shear $\mathbf{u} = (\gamma y, 0, 0)$ will stretch the surface area of a dyed region which starts as a sphere so that it grows linearly in time like $2\pi a^2 \gamma t$. (This linear growth is attained after a total shear of about $\gamma t = 3$.) The linear growth leads to a Liapounov exponent which vanishes, and this reflects the poor mixing of steady simple shear.

Now consider applying the above simple shear for a duration Δt and then switching to a second simple shear $\mathbf{u} = (0, \gamma' x, 0)$ in a different direction. The region of dyed fluid will remain ellipsoidal, and after a duration $\Delta t'$ of the second shear it will have a surface area of approximately $2\pi a^2 \gamma \Delta t \gamma' \Delta t'$, supposing that the total shears $\gamma \Delta t$ and $\gamma' \Delta t'$ are greater than 3. If this cycle is repeated several times, then after a time t the surface areas will be

$$2\pi a^2 (\gamma \Delta t \gamma' \Delta t')^{t/(\Delta t + \Delta t')}.$$

Thus we have exponential growth in time with a Liapounov exponent

$$\sigma_A = \gamma \frac{\ln \gamma \Delta t}{\gamma \Delta t}$$

in the optimal case of $\Delta t' = \Delta t$ and $\gamma' = \gamma$. At $\gamma\Delta t = 3$ this is 0.37γ . Thus we see that time-periodic two-dimensional flows might lead to useful mixing.

4.4 The blinking pair of vortices

The two infinite simple shear flows are far removed from any practical device. A very interesting step towards a realisable flow is the blinking pair of line vortices suggested by Aref (1984). Consider some dyed fluid which starts near one line vortex, and let that vortex turn while the other remains at rest. The dye will be sheared by the near vortex, and after sufficient time will be spread around the circle through its initial position. Now stop the first vortex and switch on the second. The shear flow from the second vortex will be perpendicular to the shear from the first at most locations. The dye will therefore be stretched efficiently for the first few strains. One then switches back to the first vortex and continues in a cyclic manner, hence the name ‘blinking pair of vortices’.

While the dye remains near the first vortex (whether this occurs depends on the durations of the ‘blinks’), one can see the dye stretched out by the far vortex and then wrapped round by the near vortex. This *stretching-and-folding* is an essential ingredient of good mixing in finite devices, the stretching of interfacial area being the good mixing and the folding being necessary to fit the exponentially stretching material into a finite space. This process of stretching-and-folding is well known by bakers kneading bread by hand.

Figure 7.3.8 on page JMO 184 shows how an initial line of points spread nearly uniformly within a large region after 20 iterations of the blinking pair of vortices. While the mixing is good within this region, which encloses the initial line and circles of similar dimensions around the two vortices, it is striking that there is no mixing out of the region. Figure 7.3.11 on page JMO 188 (by Khakhar, Rising and Ottino 1986) shows with twice the frequency of blinking that most of the material starting near one vortex remains near that vortex for 25 iterations. There is thus poor communications between domains of the two vortices under certain circumstances. These two figures do illustrate that while Lagrangian chaos can produce nearly uniform mixing within part of the flow it does also suffer problems of not achieving good global mixing.

4.5 Alternating eccentric cylinders

A practical realization of mixing with cyclic switching between two shears in different directions is a fluid in the annular gap between two eccentrically mounted cylinders which rotate alternately around their fixed axes. Figure 7.4.5 on page JMO 153g shows the positions visited at the end of each of 1000 iterations of 8 particles using a cycle in which the outer cylinder is turned through a half turn and then the inner cylinder is turned backwards through a whole turn. Again one notes good uniform mixing in a large area, but roughly an equal area which is not visited and is unmixed.

4.6 Alternating helical mixers

Another practical device for efficient mixing based on the idea of alternating the direction of the flow is Kenics’ static mixer, figure 8.2.2 on page JMO 223. This device consists of a circular pipe with a sequence of central dividers which turn helically along the pipe, in one direction for one turn, to be followed down the pipe by the next turning in the

opposite direction and starting perpendicular to the preceding divider. The flow is thus split in two, twisted and then recombined a number of times. An interface will thus grow exponentially downstream after undergoing several cycles. Idealised studies of the device (without the helices), described in §8.2 of JMO, find thorough mixing in some parts of the flow, together with poorly mixed regions.

4.7 One origin of chaos

This section becomes quite complicated and can be safely skipped by those interesting in mixing.

While steady two-dimensional flows cannot have chaotic fluid pathlines, time-periodic two-dimensional flows can. We consider in this section a two-dimensional flow with a small amplitude periodic variation about a steady base flow.

Almost all of the streamlines in the base flow are closed. The speed along each closed streamline will have a non-zero minimum. As long as the periodic perturbation flow is small compared with this minimum, the fluid pathlines will remain near to the streamlines of the base flow, and hence rather uninteresting.

The interesting streamlines of the base flow are the very few which go to or leave the stagnation points, ‘hyperbolic points’ in the language of dynamical systems. At a stagnation point the velocity of the base flow vanishes, and so small perturbations can exert a controlling influence on pathlines passing near it.

A special feature of incompressible steady two-dimensional flows is that a streamline that leaves one stagnation must go precisely to another, or re-approach itself. If this were not the case, fluid would enter a region never to exit, so violating conservation of fluid volume.

Now as the flow approaches a stagnation point it divides into two streams, each stream passing on a different side of the stagnation point. The streamlines joining stagnation points thus separate the flow into non-communicating vortices, and for this reason are called ‘separatrices’.

We now consider the effect of small time-periodic perturbations on the separatrices of the base flow. The effect of the perturbation can be represented by plotting the sequence of positions of a fluid blob at the end of each period, the so called ‘Poincaré section’ for the flow. Consider a collection of sequences which start from a short line perpendicular to the separatrix at some position well away from the stagnation point. Some of these sequences will be deflected to one side of the stagnation point and some to the other. There must however be one starting point along the short line whose sequence is deflected neither side and instead slowly approaches a fixed point in the Poincaré section very near to the stagnation point of the base flow. By considering short lines at different places along the separatrix, one can construct a new separation line in the Poincaré section. Fluid particles do not remain on this line during the periodic perturbations, but return to it at the end of each period. Again the new separation line separates fluid (at the start of each period) which will divide and go off around vortices differently.

An important difference between a time-periodic incompressible two-dimensional flow and a steady one is that a separation line leaving one fixed point in the Poincaré section does not have to go to another fixed point, or to re-approach itself. The conservation of fluid area only requires balancing areas go each side of the second fixed point during the period. Thus the separation line leaving one fixed point must cross the different separation line which approaches the second fixed point. If it does cross once, then we

can conclude that there must be an infinite number of crossings as one goes along the line to each fixed point, because the sequence of points starting from the first crossing must lie by definition on both lines as one goes along the lines to their fixed end points..

Thus in any small neighbourhood of one fixed point in the Poincaré section, the separation line leaving the fixed point is crossed an infinite number of times by the separation line which approaches the following fixed point. This means that lying on one side of the separation line leaving the first fixed point there is an infinite sequence of points exiting the following fixed point on alternating sides. This means that approaching the first fixed point one can find a point closer to the entering separation line which exits the following fixed point on either side.

This argument can be immediately applied to the fixed point following the following one. Hence one can select any arbitrary route past connected fixed points, diverting either side at each, by moving slightly nearer to the initial separating line. This random route is chaos, at least in the neighbourhood of the separatrices of the original steady two-dimensional flow.

4.8 Observations (critical)

Lagrangian chaos is a young subject but has already made a major impact on the way we think about mixing. Lagrangian chaos leads to the possibility of efficient mixing at low Reynolds numbers, and leads to a better understanding of turbulent mixing when the diffusivity is small ($D \ll \nu$).

Despite many papers published recently on Lagrangian chaos, there is little quantitative understanding. For example, there is no theory for the value of the Liapounov exponents, and we cannot predict how much of the flow is poorly mixed and where it will be.

Some of the fascinating details of islands of chaos around islands around islands may be rapidly obliterated by the addition of a little molecular diffusion. In fact the transfer across ‘barriers’ between adjacent well-mixed regions, whether by molecular diffusion or by non-periodic fluctuations, is a very important issue in several applications and yet something we have hardly started to understand. Perhaps time-periodic two-dimensional flows have been over-studied and now three-dimensional and non-periodic flows deserve greater attention.

Above all, the engineering questions must not be lost sight of. How much mixing is there after just 5 cycles, not the theoretician’s 1000? Open flow devices with a continuous throughput are essential in industry. What is their mixing efficiency, what is the residence time of the fluid, and of course what is the cost?

5 References

- M. Van Dyke *An Album of Fluid Motion* Parabolic Press (1982).
J.M. Ottino *The kinematics of mixing: stretching, chaos, and transport* C.U.P. (1989).