Mathematical Tripos Part IA — E.J. Hinch
Dynamics (C8) — February 1998

Example Sheet 4

1. *System, Virial.* A system of point particles of masses m_α at positions $\mathbf{r}_\alpha$ move under interparticle forces with a potential energy $V = -\sum_{\alpha>\beta} A_{\alpha\beta}|\mathbf{r}_\alpha - \mathbf{r}_\beta|^{-n}$. What is the force on particle α? Let $J = \frac{1}{2}\sum_\alpha m_\alpha \mathbf{r}_\alpha^2$. Show that $d^2J/dt^2 = 2K + nV$, where K is the total kinetic energy.

2. *System, internal motion.* The triatomic molecule CO_2 consists of 3 atoms of masses m, m' and m' at positions $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. Express these positions in terms of the three variables $\mathbf{R}$, the centre of mass, $\mathbf{P} = \mathbf{r}_1 - \mathbf{R}$ and $\mathbf{Q} = \mathbf{r}_2 - \mathbf{r}_3$. Now show that $\mathbf{P}(t)$, $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ make separate contributions to the total kinetic energy and total angular momentum.

3. *System, CofM motion.* A particle of mass m_1 at rest is hit by a particle of mass m_2 moving at velocity $\mathbf{v}$. The collision produces N fragments of equal mass, each having the same energy relative to the centre of mass. Assuming that energy is conserved, show that the fragments lie on a sphere whose centre moves with velocity $m_2\mathbf{v}/(m_1+m_2)$ and whose radius expands with speed $\sqrt{m_1 m_2}|\mathbf{v}|/(m_1+m_2)$.

4. *Reduced mass.* Two protons of mass m and electric charge e approach one another in the same plane at the same speed V from directions differing by $\pi - 2\beta$. Without electrostatic interactions the closest approach would be b. Show that the actual closest approach is the positive root of the equation

$$x^2 \;-\; \frac{e^2}{4\pi\epsilon_0 m V^2\cos^2\beta}x \;-\; b^2 \;=\; 0\,.$$

5. *Moments of inertia.* Find the moments of inertia of a solid cylinder of length $2l$ and radius a (i) about its axis, and (ii) about a perpendicular axis through its centre. Similarly find the moments of inertia of a solid circular cone of height h and radius of the base a (i) about its axis, and (ii) about a perpendicular axis through the apex. What is the period of small oscillations of a pendulum made of a solid cone suspended from its apex?

6. *Translation and rotation.* A yo-yo consists of two heavy disks of mass M and radius R connected by a light axle of radius a, around which one end of a string is wound, the other end being held fixed. Assuming that the unwound part of the string is approximately vertical, how fast does the yo-yo accelerate downwards?

Please notify all errors to `E.J.Hinch@damtp.cam.ac.uk`.

7. *Translation and rotation.* A sphere of mass M is thrown along a rough horizontal table at speed V, initially with no angular velocity. While the sphere slips, there is a frictional force $\mu M g$. How far does the sphere travel before its starts to roll without slipping? (Could dimensional analysis be usefully applied here?) What is the eventual rolling speed?

8. *Percussion.* Approximate a baseball bat by a solid circular cone held at its apex. When a ball is struck, there is an impulsive change to the translational and rotational motion (use CofM). How far along the bat should the ball be struck in order that the impulsive change in the velocity of the apex be zero?

9. *Conservations, difficult.* A solid sphere of radius a rolls along a level surface and collides with a step of height h. Show that the minimum velocity for the sphere to climb the step, assuming no slipping at the point of contact with the step, is $V = \sqrt{\frac{10}{7}gh}\Big/(1-\frac{5}{7}h/a)$. [*Hint:* find the total angular momentum of the sphere about the point of contact just prior to contact; this is conserved during the collision; find the angular velocity about the fixed point of contact just after the collision; hence the kinetic energy then.]

10. *Torques.* A car with a transverse engine moves at speed V around a corner of radius R. What is the direction and magnitude of the rate of change of angular momentum L of the rotating part of the engine? In what sense should the engine rotate in order to assist the cornering (couple applied by the engine on the car counters the couple about the centre of mass of the force from the road required to go around corner)? Discuss longitudinally mounted engines and bicycles.

11. *Phase-planes.* Sketch the phase-plane portraits of the following equations, using linearised analysis around the equilibria where appropriate.

$$
\begin{aligned}
&(a) \qquad \ddot{x} + 4\dot{x} + x = 0 \\
&(b) \qquad \ddot{x} + \dot{x} + x = 0 \\
&(c) \qquad \ddot{x} - x + x^3 = 0 \\
&(d) \qquad \ddot{x} + \dot{x}(\dot{x}^2 + x^2 - 1) + x = 0 \\
&(e) \qquad \ddot{x} + \mathrm{sign}(\dot{x}) + x = 0\ .
\end{aligned}
$$

In (d) note the solution $x(t) = \sin t$.