Standard Model 1, P, C and T

(1) The four dimensional 4×4 Dirac matrices are defined uniquely up to an equivalence by $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}1$, with 1 the unit matrix. We may also require that if $\gamma^\mu = (\gamma^0,\boldsymbol{\gamma})$ then $\gamma^{\mu\dagger} = (\gamma^0,-\boldsymbol{\gamma})$. If $[X,\gamma^\mu]=0$ for all μ then $X\propto 1$ and if γ^μ,γ'^μ both obey the Dirac algebra then $\gamma'^\mu = S\gamma^\mu S^{-1}$ for some S. Define the charge conjugation matrix C by $C\gamma^{\mu\,t}C^{-1} = -\gamma^\mu$, where t denotes the matrix transpose. Show that $[C^tC^{-1},\gamma^\mu]=0$ and hence that $C^t = cC,\ c=\pm 1$. Derive the results

$$\big(\gamma^\mu C\big)^t = -c\gamma^\mu C\,,\quad \big(\gamma_5 C\big)^t = c\gamma_5 C\,,\quad \big(\gamma^\mu\gamma_5 C\big)^t = c\gamma^\mu\gamma_5 C\,,\quad \big([\gamma^\mu,\gamma^\nu]C\big)^t = -c[\gamma^\mu,\gamma^\nu]C\,.$$

Hence, since there are 6 independent antisymmetric and 10 symmetric 4×4 matrices, show that we must take $c=-1$.
Show also, using the assumed hermeticity properties of the Dirac matrices, $[\gamma^\mu, CC^\dagger]=0$ so that we may take $CC^\dagger = 1$.
The matrix B is defined by $B\gamma^{\mu *}B^{-1} = (\gamma^0,-\boldsymbol{\gamma})$. Show that $B\gamma_5{}^*B^{-1} = \gamma_5$. With the assumed form for $\gamma^{\mu\dagger}$ verify that we may take $B = \pm\gamma_5 C$.
*Generalise the above argument for finding c to $2n$ dimensions when the Dirac matrices are $2^n\times 2^n$ and we may take as a linearly independent basis 1 and $\gamma^{\mu_1\cdots\mu_r} = \gamma^{[\mu_1}\ldots\gamma^{\mu_r]}$, where [ ] denotes antisymmetrisation of indices, for $r=1,\ldots 2n$ ($\gamma^{\mu_1\cdots\mu_r}$ has $\binom{2n}{r}$ independent components). Show that $C(\gamma^{\mu_1\cdots\mu_r})^tC^{-1} = (-1)^{\frac12 r(r+1)}\gamma^{\mu_1\cdots\mu_r}$ and hence $c = (-1)^{\frac12 n(n+1)}$. Generalise γ_5 by taking $\hat\gamma = i^{n-1}\gamma^0\gamma^1\ldots\gamma^{2n-1}$ and show that $\hat\gamma$ is hermitian and $\hat\gamma^2=1$. Show that

$$\psi^c = C\overline{\psi}^{\,t}\,,\quad \psi' = \hat\gamma\psi \quad\Rightarrow\quad \psi = -c\,C\overline{\psi^c}^{\,t}\,,\quad \psi'^c = -(-1)^n\hat\gamma\psi^c\,.$$

In what dimensions is possible to have Majorana-Weyl spinors, so that $\psi^c = \pm\psi' = \psi$?

(2) A Dirac quantum field transforms under parity so that

$$\hat P\psi(x)\hat P^{-1} = \gamma^0\psi(x_P)\,,\quad x_P^\mu = (x^0,-\mathbf{x})\,,$$

and has an interaction with a scalar field $\phi(x)$

$$\mathcal{L}_I(x) = g\,\overline{\psi}(x)\psi(x)\phi(x) + g'\overline{\psi}(x)i\gamma_5\psi(x)\phi(x)\,.$$

How must $\phi(x)$ transform under parity, i.e. obtain the necessary form for $\hat P\phi(x)\hat P^{-1}$, to ensure that the theory is invariant under parity if $g'=0$? What are the transformation properties of $\phi(x)$ for parity invariance when $g=0$. Can parity be conserved in a theory if both g,g' are non zero.
How does the axial current $j_5^\mu(x) = \overline{\psi}(x)\gamma^\mu\gamma_5\psi(x)$ transform under parity?

(3) For a free operator Dirac field $\hat\psi(x)$ assume $\hat\psi(x) = \sum_r a_r\,\psi_r(x)$ where $\{\psi_r(x)\}$ forms a basis for solutions of the Dirac equation and a_r are operators. Explain why a basis may be chosen so that $B\psi_r(x)^* = \psi_{r'}(x_T)$ where $x_T^\mu = (-x^0,\mathbf{x})$ and $B\gamma^{\mu *}B^{-1} = (\gamma^0,-\boldsymbol{\gamma})$. Assume the time reversal operator is defined so that $\hat T a_r\hat T^{-1} = a_{r'}$. What is $\hat T\hat\psi(x)\hat T^{-1}$?

(4) Under charge conjugation and time reversal a Dirac field ψ transforms as

$$\hat C\psi(x)\hat C^{-1} = C\overline{\psi}(x)^t\,,\qquad \hat T\psi(x)\hat T^{-1} = B^{-1}\psi(x_T)\,,\quad x_T^\mu = (-x^0,\mathbf{x})\,.$$

with $\hat{C}$, $\hat{T}$ the unitary, anti-unitary operators implementing these operations (note, if $\hat{T}|\phi\rangle = |\phi_T\rangle$ then $\langle\phi'|\phi\rangle = \langle\phi_T|\phi'_T\rangle$). The matrices C, B are defined in question 1 with also $C^\dagger C = B^\dagger B = 1$. Show that, if X is matrix acting on Dirac spinors,

$$\hat{C}\,\overline{\psi}(x)X\psi(x)\hat{C}^{-1} = \overline{\psi}(x)X_C\psi(x)\,, \quad \hat{T}\,\overline{\psi}(x)X\psi(x)\hat{T}^{-1} = \overline{\psi}(x_T)X_T\psi(x_T)\,,$$

where $X_C = CX^tC^{-1}$ (take ψ and $\overline{\psi}$ to anti-commute) and $X_T = BX^*B^{-1}$. Hence determine the transformation properties under charge conjugation and time reversal of

$$\overline{\psi}(x)\psi(x)\,, \qquad \overline{\psi}(x)i\gamma_5\psi(x)\,, \qquad \overline{\psi}(x)\gamma^\mu\gamma_5\psi(x)\,.$$

If $|p\rangle$ is a boson with momentum p and $\langle 0|\overline{\psi}(0)i\gamma_5\psi(0)|p\rangle \neq 0$ show that, in a theory in which P, C are conserved, then the boson must have negative intrinsic parity and also positive charge conjugation parity.

(5) From Maxwell's equation $\partial_\nu F^{\mu\nu} = e\overline{\psi}\gamma^\mu\psi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ derive the required transformation properties of $A_\mu(x)$ to ensure invariance under parity, charge conjugation and time reversal. Show that $\int \mathrm{d}^4x\, \epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho}$ is odd under both P and T.

(6)* For a spin half particle of mass M the matrix element of the electromagnetic current j^μ_{em} between states of momentum p, p', $p^2 = p'^2 = M^2$ has the general form

$$\langle p', s'|j^\mu_{\text{em}}(0)|p, s\rangle = \overline{u}(p', s')\Lambda^\mu(q)u(p, s)\,, \quad q = p' - p\,,$$

with $u(p, s)$ a Dirac spinor satisfying $\gamma{\cdot}p\,u(p, s) = Mu(p, s)$. Show that current conservation, $\partial_\mu j^\mu_{\text{em}} = 0$, requires the condition $q_\mu\langle p', s'|j^\mu_{\text{em}}(0)|p, s\rangle = 0$ (use $j^\mu_{\text{em}}(x) = e^{iP\cdot x}j^\mu_{\text{em}}(0)e^{-iP\cdot x}$ for P_μ the four momentum operator) and hence that in general we may write, from Lorentz invariance,

$$\Lambda^\mu(q) = \gamma^\mu F_1(q^2) + \frac{1}{2M}i\sigma^{\mu\nu}q_\nu F_2(q^2) + \frac{1}{2M}\gamma_5\sigma^{\mu\nu}q_\nu F_3(q^2) + \Big(\gamma^\mu - q^\mu\,\frac{2M}{q^2}\Big)\gamma_5 F_A(q^2)\,,$$

where $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], {\gamma_5}^\dagger = \gamma_5$. Show that hermeticity, ${j^\mu_{\text{em}}}^\dagger = j^\mu_{\text{em}}$, requires $\gamma^0\Lambda^\mu(-q)^\dagger\gamma^0 = \Lambda^\mu(q)$ and hence that $F_{1,2,3,A}(q^2)$ are real. Verify that we may take $u(p_P, s) = \gamma^0u(p, s)$ and $u(p_P, -s)^*\eta_s = B^{-1}u(p, s)$, for some phase η_s. Use the standard transformation properties of the current, j^μ_{em}, and the states, $\hat{P}|p, s\rangle = |p_P, s\rangle$, $\hat{T}|p, s\rangle = |p_P, -s\rangle\eta_s$, to show that invariance under parity requires Λ to satisfy $\gamma^0\Lambda^\mu_P(q_P)\gamma^0 = \Lambda^\mu(q)$ and for time reversal invariance $B\Lambda^\mu_P(q_P)^*B^{-1} = \Lambda^\mu(q)$ (where $\Lambda^\mu_P = (\Lambda^0, -\boldsymbol{\Lambda})$). Hence show that both discrete symmetries require $F_3(q^2) = 0$ and parity $F_A(q^2) = 0$.

(7) For a Dirac field ψ, with $\bar{\psi} = \psi^\dagger\gamma^0$, the Lagrangian is

$$\mathcal{L} = \bar{\psi}\,i\gamma{\cdot}\partial\,\psi - \bar{\psi}\,M\,\psi + \tfrac{1}{2}\,\psi^t\,C^{-1}m\,\psi - \tfrac{1}{2}\,\bar{\psi}\,\bar{m}\,C\,\bar{\psi}^t\,,$$

with M, m in principle matrices satisfying $M = \gamma^0M^\dagger\gamma^0$, $\bar{m} = \gamma^0m^\dagger\gamma^0$, $C^{-1}mC = m^t$. Show that $\mathcal{L}^\dagger = \mathcal{L}$. Obtain the equation

$$(i\gamma\cdot\partial - \mathcal{M})\Psi = 0\,, \qquad \Psi = \begin{pmatrix} \psi \\ C\bar{\psi}^t \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} M & \bar{m} \\ m & M \end{pmatrix}.$$

Assume that the physical masses m_p are determined by $p^2 = m_p{}^2$ where $\det D_p = 0$ for the Dirac operator $D_p = p\cdot\gamma - \mathcal{M}$. By considering $\det(\gamma_5D_p\gamma_5D_p) = \det(D_{-p}D_p)$ show that, if M, m have just one component, the physical masses are $|M \pm |m|\,|$.