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Weyl Consistency Conditions and a Local Renormalisation Group Equation for General Renormalisable Field Theories

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A local renormalisation group equation which realises infinitesimal Weyl rescalings of the metric and which is an extension of the usual Callan-Symanzik equation is described. In order to ensure that any local composite operators, with dimensions so that on addition to the basic lagrangian they preserve renormalisability, are well defined for arbitrarily many insertions into correlation functions the couplings are assumed to depend on x. Local operators are then defined by functional differentiation with respect to the couplings just as the energy momentum tensor is given by functional differentiation with respect to the metric. The local renormalisation group equation contains terms depending on derivatives of the couplings as well as the curvature tensor formed from the metric, constrained by power counting. Various consistency relations arising from the commutativity of Weyl transformations are derived, extending previous one loop results for the trace anomaly to all orders. In two dimensions the relations give an alternative derivation of the c-theorem and similar extensions are obtained in four dimensions. The equations are applied in detail to general renormalisable σ models in two dimensions. The Curci-Paffuti relation is derived without any commitment to a particular regularisation scheme and further equations used to construct an action for the vanishing of the β functions are also obtained. The discussion is also extended to σ models with a boundary, as appropriate for open strings, and relations for the additional β functions present in such models are obtained.

1. Introduction

In principle the requirements of Weyl symmetry, or invariance under local rescaling of the spatial metric $\gamma_{\mu\nu} \to e^{-2\sigma}\gamma_{\mu\nu}$ for arbitrary $\sigma(x)$, are much stronger than simple scale invariance where σ is taken to be a constant. For Weyl invariance the trace of the conserved symmetric energy momentum tensor $T_{\mu\nu}$ vanishes, which is a local equation, whereas for scale invariance it is only necessary that the trace is a total divergence. When a Weyl invariant theory is restricted to a flat metric, assuming that it is also invariant under diffeomorphisms, then the theory is symmetric under the full conformal group, which of course is infinite dimensional in two dimensions, whereas scale or dilation invariance by itself adds just one additional generator to the Lorentz or Poincaré algebra. However in many simple field theories scale invariance is sufficient to imply symmetry under the conformal group as well [1]. Assuming any couplings with dimensions, or generalised mass terms, are absent the requirements for such an invariance in the quantum theory are then identified with the vanishing of the β function for all dimensionless couplings. This condition ensures that physical amplitudes or correlation functions in the quantum field theory are independent of the renormalisation mass scale μ.

As an instance of the possibility of scale invariance implying Weyl symmetry if a field theory is such that, after applying the equation of motion and restricting the flat space, the operator equation

$$\gamma^{\mu\nu}T_{\mu\nu} = \Theta = \beta^i \mathcal{O}_i \ , \tag{1.1}$$

is valid for some basis of scalar operators $\mathcal{O}_i(x)$ associated with couplings g^i for which the β functions are β^i, then manifestly $\beta^i = 0$ implies also $\gamma^{\mu\nu}T_{\mu\nu} = 0$ as required for Weyl symmetry (the conserved currents whose charges generate the conformal group are $J_\xi^\mu = T^{\mu\nu}\xi_\nu$ where $\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = \gamma_{\mu\nu}\,\lambda,\ d\lambda = 2\nabla^\mu\xi_\mu$ for dimension d). On curved space there are generally additional curvature dependent terms on the r.h.s. of (1.1). In two dimensions, for renormalisable theories, these are proportional to just the scalar curvature R and when $\beta^i = 0$ the coefficient is, up to a factor, just the central charge c of the Virasoro algebra.

Nevertheless since the β functions indicate only the response of the couplings g^i to constant rescalings of the renormalisation scale μ we may expect generically instead of (1.1)

$$\Theta = \beta^i \mathcal{O}_i + \nabla_\mu \mathcal{Z}^\mu + \text{curvature dependent terms} \ , \tag{1.2}$$

with $\mathcal{Z}^\mu(x)$ a local current. On flat space with $\beta^i = 0$ this still implies scale invariance (the conserved current whose charge generates dilations is $T^{\mu\nu}x_\nu - \mathcal{Z}^\mu$). If $\mathcal{Z}^\mu = \nabla_\nu L^{\mu\nu}$ for some operator $L^{\mu\nu} = L^{\nu\mu}$ then a local redefinition of $T_{\mu\nu} \to T'_{\mu\nu}$ maintaining $T'_{\mu\nu} = T'_{\nu\mu}$, $\nabla^\mu T'_{\mu\nu} = 0$ is possible which cancels the $\mathcal{Z}^\mu$ term in (1.2) (for $d = 2$ $\mathcal{Z}^\mu = \nabla^\mu \mathcal{S}$ is necessary) [2,3]. Hence in this case conformal invariance is still valid when $\beta^i = 0$ with the charges expressed in terms of $T'_{\mu\nu}$. On the other hand $\mathcal{Z}^\mu$ may be a current associated with the generators of a symmetry G and by using the corresponding operator conservation equation, involving the equations of motion, $\nabla_\mu \mathcal{Z}^\mu$ may be expressible in terms of the basis of scalar operators $\mathcal{O}_i$. This then induces a redefinition of the β functions $\beta^i \to B^i = \beta^i + \delta^G_{\mathcal{Z}} g^i$ where $\delta^G g^i$ denotes the action of some infinitesimal symmetry variation on the coupling g^i, with $\delta^G_{\mathcal{Z}}$ determined by the current $\mathcal{Z}^\mu$ [4]. Effectively in this situation the β functions are arbitrary, $\beta^i \sim \beta^i + \delta^G g^i$, so that the condition for scale invariance is then $\beta^i \sim 0$ [5] whereas for Weyl invariance the necessary condition is that $B^i = 0$ in which this freedom cancels.

A crucial constraint on the allowed form of the trace of the energy momentum tensor follows from the requirement that $\langle \Theta(x) \rangle$ is determined by the response of the vacuum self energy functional W to an infinitesimal Weyl rescaling, $\delta_\sigma W$, of the metric and should therefore satisfy an appropriate integrability condition derived from the fact that the group of Weyl transformations is abelian, $(\delta_\sigma \delta_{\sigma'} - \delta_{\sigma'} \delta_\sigma) W = 0$. Indeed for classically Weyl invariant theories such consistency conditions, analogous to those which constrain the axial anomaly, have been applied in 2, 4 and 6 dimensions to determine the possible form of the curvature terms in (1.2) [6,7]. In such models $\Theta = 0$ classically but at one loop in the quantum theory there is a local Weyl anomaly given by the various possible scalars of dimension d formed from the curvature tensor and its covariant derivatives. Of course $\langle \Theta(x) \rangle$ is arbitrary up to terms which arise from additional finite contributions to W given by integrals over local scalars of dimension d formed from the Riemann tensor and its covariant derivatives. Thus the possible form of the Weyl anomaly becomes a cohomological problem. In four dimensions the consistency condition implies that there are no R^2 terms, as opposed to terms proportional to the square of the Weyl tensor F or the Euler density G, and further any $\nabla^2 R$ terms are trivial in the sense that they can be removed by a local redefinition [6,7].

Beyond one loop $\langle \Theta(x) \rangle$ is no longer a local functional of the metric and its derivatives at x and the consistency conditions are not immediately obvious. In this paper $\mathcal{A} = \langle \Theta \rangle - \beta^i \langle \mathcal{O}_i \rangle - \nabla_\mu \langle \mathcal{Z}^\mu_{\text{op}} \rangle$, where $\mathcal{Z}^\mu_{\text{op}}$ is the operator part of $\mathcal{Z}^\mu$, is assumed to remain a local expression to all orders allowing therefore an extension of the usual arguments. In order to apply the discussion to $\mathcal{A}$ it is necessary to be precise about defining $\mathcal{O}_i$, and also $\mathcal{Z}^\mu_{\text{op}}$, as finite local composite operators. To achieve this we adopt the trick [8,9] of allowing all the couplings g^i to be arbitrary functions of x and, assuming W is still determined as a finite functional by the regularised quantum field theory to all orders, correlation functions of arbitrarily many finite local operators $\mathcal{O}_i(x)$ at non coincident points may then be defined by functional differentiation with respect to $g^i(x)$, just as insertions of $T_{\mu\nu}(x)$ are given by functional derivatives with respect to the metric $\gamma^{\mu\nu}(x)$. In this case $\mathcal{A}$ is a local function involving the metric and also derivatives $\partial_\mu g^i$. Such terms are relevant, even for g^i constant, when considering correlation functions of products of the operators $\mathcal{O}_i$ and are directly related to the additional divergences present in such cases beyond those removed in the definition of $\mathcal{O}_i$ itself as a finite local operator. The results include non trivial relations between the purely curvature dependent terms in $\mathcal{A}$ and those involving $\partial_\mu g^i$ which in principle allow the former to be determined by purely flat space calculations (save for the F term in 4 dimensions). Thus the R^2 term in $\mathcal{A}$ in 4 dimensions is determined and is non zero at sufficiently high order.

Such results were first obtained by Brown and Collins in scalar ϕ^4 field theory using dimensional regularisation although the derivation is seemingly very different and depended quite strongly on the particular regularisation scheme [10]. Their work was extended, also in the framework of dimensional regularisation, to include the G term by Hathrell [11] and also to gauge theories by Hathrell and Freeman [12]. More recently the essential equations of this paper, which extend the work of Brown and Collins, Hathrell and Freeman to a more general situation with many couplings, were obtained in the context of dimensional regularisation and using the idea of x dependent couplings in two dimensions, in particular for general non linear σ models [13], and four dimensions [9,14]. In both cases detailed calculations have been made of the various new coefficients introduced at two or more loops using dimensional regularisation [8,13,14,15]. The consistency conditions derived in this paper give identical equations but without any commitment to a particular regularisation scheme. This should allow extensions to cases, such as heterotic σ models, where there are anomalies in some symmetries and dimensional regularisation becomes problematic. The starting point is an equation expressing the essential content of (1.2) where the curvature dependent includes now also an arbitrary dependence on $\partial_\mu g^i$ consistent with the usual symmetries and each term having an overall dimension d. This is in effect a local version of the Callan-Symanzik equation and from which consistency conditions necessary for the integrability of W can readily be computed. As with the Callan-Symanzik equation the basic equations may presumably be derived within the loop expansion in a well defined regularisation scheme but should also be valid non perturbatively.

In section 2 the consistency relations are obtained in a simplified two dimensional model containing only dimension two operators. The resulting equations are essentially those at the basis of Zamolodchikov's c-theorem [16] and in this section it is shown how the crucial positive definite metric on the space of couplings may be obtained by adding a cohomologically trivial (from the point of view of Weyl scaling) terms to a symmetric tensor χ_{ij} defined in terms of the local Callan-Symanzik equation. Alternatively, even on flat space and with constant couplings, χ_{ij} is determined by the inhomogeneous term, proportional to $\partial^2\delta^2(x)$, in the conventional Callan-Symanzik equation for the two point correlation function $\langle\mathcal{O}_i(x)\mathcal{O}_j(0)\rangle$ which is symmetric and positive definite and, after removing the leading singularity x^{-4} as $x \to 0$, serves to define Zamolodchikov's metric. The extension of the consistency relations to two dimensional theories defined on a space with a boundary is also discussed. In this case there are additional boundary contributions to the local renormalisation group equation which involve the extrinsic curvature K.

In four dimensions the full set of consistency relations, which are derived in section 3, become rather complicated although they include equations of essentially the same form as those defining the c-theorem in two dimensions with the c-function given in terms of the coefficient of the G term in the energy momentum tensor trace, but without the corresponding tensor to χ_{ij} being related to a manifestly positive correlation function and so defining a metric as yet. This possible form for a four dimensional c-function was first suggested by Cardy [17] and has subsequently been analysed in a perturbative framework [14]. Various other possibilities for a four dimensional c-theorem have recently been discussed by Capelli *et al* [18]. We also show how to obtain a result inspired by the c-theorem which has been recently derived by Shore [19] using a spectral representation of the two point function for the energy momentum tensor in which positivity properties are clear cut. The resulting equation expresses the behaviour of a function of the couplings, which is related to the coefficient of the F term in th energy momentum tensor trace in four dimensions, under a change of scale, as controlled by the renormalisation group. However this equation contains an additional term away from two dimensions so that the crucial monotonic flow of the c-function as originally defined by Zamolodchikov is no longer obtained. In order to extend the analysis of consistency relations to realistic field theories the modifications of the consistency relations arising from the presence of lower dimension operators, such as are present in general renormalisable field theories in four dimensions, are also discussed.

In section 4 the consistency conditions are applied to general bosonic σ models in two dimensions. These are of interest in string theory since the vanishing of the β function may be identified with the equations of motion for the low energy degrees of freedom [20], generalising the Einstein equations for gravity, and the analysis is complicated by the presence of lower dimension operators such as those representing the tachyon. The c-function is now in general a scalar operator which leads to an extra piece in the equation for the renormalisation flow of the c-function arising from the anomalous dimension of the scalar operator, only if the equation can be restricted to the dimension zero c-number part of the operator is the simple form of Zamolodchikov's equation obtained. However the resulting equations include the Curci-Paffuti relation [21] which shows how the dilaton β function β^{Φ}, or rather B^{Φ}, becomes a constant at the conformal point and also another equation which determines how B^{Φ} depends on the target space metric G_{ij} and antisymmetric tensor B_{ij} which are the essential couplings in these σ models. This equation was used by us previously, subject to some caveats, to construct an action to all orders whose variation vanishes when $B^G_{ij} = B^B_{ij} = 0$ [13]. This action generalises a suggestion of Tseytlin [22] to a form which allows local redefinitions of G_{ij}, B_{ij}. Related ideas were discussed by Forge [23] who introduced into the couplings G_{ij}, B_{ij} a dependence on an extra field $\phi^0(x)$, in addition to those fields $\phi^i(x)$ parameterising the target manifold, so that Weyl symmetry is part of an enlarged diffeomorphism group on the extended set of coordinates. This in effect mimics the arbitrary x dependence required for our discussion and may allow for further geometrical insight.

In section 5 the analysis of the consistency relations is extended to σ models with a boundary, where the additional terms present in a boundary contribution to the action correspond to the tachyon and massless vector field present in the open string. The results provide relations between the open string and closed string β functions. Some issues concerned with boundary conditions for such general σ models are discussed in an appendix.

Finally in a conclusion a few remarks on possible extensions of the results obtained in this paper are given.

2. Two Dimensional Field Theories

The analysis of consistency relations is significantly simpler for two dimensional theories. We consider here an idealised renormalisable field theory for fields ϕ, characterised by an action S which is classically conformally invariant and depends on a set of couplings g^i corresponding to a complete set of dimension two local scalar operators $\mathcal{O}_i$. For the discussion in this section we assume the absence of any lower dimension operators.

In order to derive the full set of consistency conditions for the associated quantum field theory the couplings g^i are assumed to be arbitrary functions of x, so that they also play the role of sources for the operators $\mathcal{O}_i$, and S is further required to be defined for a general curved background spatial metric $\gamma_{\mu\nu}$, with the lagrangian density a scalar under reparameterisations of x. Hence we may define

$$\mathcal{O}_i(x) = \frac{\delta}{\delta g^i(x)} S \ , \qquad T_{\mu\nu}(x) = 2 \frac{\delta}{\delta \gamma^{\mu\nu}(x)} S \ , \tag{2.1}$$

for $T^{\mu\nu}$ the energy momentum tensor, under the equations of motion $\nabla_\mu T^{\mu\nu} = 0$ and classically $\gamma_{\mu\nu} T^{\mu\nu} = 0$.

As usual in the corresponding quantum field theory $S \to S_0$ where S_0 includes all necessary counterterms of dimension two or less in some definite regularisation prescription. In this case the counterterms, besides those obtained by $g^i \to g_0^i(g)$, depend on $\partial_\mu g^i$ and, assuming the the regularisation scheme preserves manifest covariance under reparameterisations of the coordinates x, also on the scalar curvature R formed from the metric $\gamma_{\mu\nu}$. S_0 may include additional couplings associated with operators with dimensions > 2 introducing a cut off scale Λ and which are a necessary part of the regularisation procedure but the couplings for such irrelevant operators are supposed to be determined in terms of the finite couplings g^i by the requirement that correlation functions and hence physical amplitudes are independent of Λ as $\Lambda \to \infty$. Any such correlation function may be expressed in terms of the vacuum energy functional W by functional differentiation with respect to appropriate sources. W is defined by

$$e^W = \int d[\phi] \, e^{-\frac{1}{\ell} S_0} \ , \tag{2.2}$$

where ℓ is a loop counting parameter. The functional derivatives of W with respect to g^i and $\gamma^{\mu\nu}$ then give the connected correlation functions of the finite local quantum operators $[\mathcal{O}_i] = \delta S_0 / \delta g^i$ and the quantum energy momentum tensor $T_{\mu\nu} = 2\delta S_0 / \delta \gamma^{\mu\nu}$.

For a local Weyl rescaling $\delta\gamma^{\mu\nu} = 2\sigma\gamma^{\mu\nu}$ we define

$$\begin{aligned} \Delta_\sigma^W &= 2 \int dv \, \sigma \, \gamma^{\mu\nu} \frac{\delta}{\delta \gamma^{\mu\nu}} \ , \qquad dv = d^2 x \sqrt{\gamma} \ , \\ \Delta_\sigma^\beta &= \int dv \, \sigma \, \beta^i \frac{\delta}{\delta g^i} \ , \end{aligned} \tag{2.3}$$

where $\beta^i(g)$ is the beta function for the coupling g^i. For classically conformally invariant theories of the form described here the operator equation $\gamma^{\mu\nu} T_{\mu\nu} = \Theta \equiv \beta^i [\mathcal{O}_i]$, relating the non zero β functions at one and higher loops to the breaking of scale invariance, is usually presumed to follow from a careful treatment of composite operators in a well defined regularisation scheme. However acting on the vacuum energy functional W this is assumed to be extended, for a general background metric and arbitrary $g^i(x)$, to

$$\Delta_\sigma^W W = \Delta_\sigma^\beta W - \frac{1}{\ell} \int dv \, \sigma \big(\tfrac{1}{2} \, \beta^\Phi R - \tfrac{1}{2} \, \chi_{ij} \partial_\mu g^i \partial^\mu g^j \big) + \frac{1}{\ell} \int dv \, \partial_\mu \sigma \, w_i \partial^\mu g^i \ , \tag{2.4}$$

which is in effect a local version of the renormalisation group equation. In general $\Delta^W_\sigma W = -\int dv\, \sigma \langle\Theta\rangle/\ell$ where $\langle\Theta(x)\rangle$ does not just depend locally on the fields at x but the content of (2.4) is that the non local contributions are given entirely by $\beta^i\langle[\mathcal{O}_i(x)]\rangle$ and the remaining part is restricted to just the form prescribed by the integrals over local expressions of dimension 2 involving σ and $\partial_\mu\sigma$. In a more general context we are tacitly also assuming a gap in dimension between c numbers of dimension zero and other possible operators in the field theory and further that there is no need for any separate infra red cut off even on restriction to flat space. The terms $\propto \sigma$ have just the same form as the additional counterterms necessary for curved space and x dependent couplings g^i. Of course for a classical scale invariant theory the r.h.s. of (2.4), without the $\Delta^W_\sigma W$ term, is the quantum scale invariance anomaly, $\Theta = \frac{1}{2}\beta^\Phi R$, at one loop on curved space which has a local form to this order. The essential assumption here is that (2.4) expresses the appropriate generalisation to all orders in the loop expansion with $\beta^\Phi(g)$, $\chi_{ij}(g)$, $w_i(g)$ perturbatively calculable in a well defined mass independent regularisation scheme. Equivalent equations of the form (2.4) have been obtained by us earlier using dimensional regularisation. For no dimensional couplings we may also impose the requirement

$$\Big(\mu\frac{\partial}{\partial\mu} + 2\int dv\, \gamma^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}}\Big)W = 0\ , \tag{2.5}$$

where μ is the arbitrary mass scale introduced in the process of renormalisation. Hence for σ constant, and on flat space with constant couplings g^i, (2.4) reduces to the conventional homogeneous Callan-Symanzik equation.

The consistency relations then follow by commuting two different local scale transformations (using for $\delta\gamma^{\mu\nu} = 2\sigma\gamma^{\mu\nu}$, $\delta R = 2\sigma R + 2\nabla^2\sigma$)

$$\begin{gathered} 0 = \big[\Delta^W_\sigma - \Delta^\beta_\sigma, \Delta^W_{\sigma'} - \Delta^\beta_{\sigma'}\big]W = \frac{1}{\ell}\int dv\, (\sigma'\partial_\mu\sigma - \sigma\partial_\mu\sigma')V^\mu\ , \\ V_\mu = \partial_\mu\beta^\Phi - \chi_{ij}\partial_\mu g^i\beta^j + \beta^j\frac{\partial}{\partial g^j}\big(w_i\partial_\mu g^i\big)\ . \end{gathered} \tag{2.6}$$

Since $g^i(x)$ is arbitrary the condition $V_\mu = 0$ becomes

$$\partial_i\beta^\Phi = \chi_{ij}\beta^j - \mathcal{L}_\beta w_i\ , \qquad \mathcal{L}_\beta w_i = \beta^j\partial_j w_i + \partial_i\beta^j w_j\ , \tag{2.7}$$

where $\mathcal{L}_\beta$ denotes the Lie derivative defined by the vector field β^i. From (2.7)

$$\partial_i\tilde\beta^\Phi = \chi_{ij}\beta^j + (\partial_i w_j - \partial_j w_i)\beta^j\ , \qquad \tilde\beta^\Phi = \beta^\Phi + w_i\beta^i\ , \tag{2.8}$$

and hence

$$\beta^i\partial_i\tilde\beta^\Phi = \chi_{ij}\beta^i\beta^j\ . \tag{2.9}$$

Of course W is arbitrary up to the addition of local functionals of the fields and the couplings. If

$$\delta W = \frac{1}{\ell}\int dv\, \big(\tfrac{1}{2}bR - \tfrac{1}{2}c_{ij}\partial_\mu g^i\partial^\mu g^j\big)\ , \tag{2.10}$$

then, for $b(g)$, $c_{ij}(g)$ an arbitrary scalar, tensor respectively on the space of couplings,

$$\begin{aligned} \delta\beta^\Phi &= \beta^i\partial_i b \ , & \delta\chi_{ij} &= \mathcal{L}_\beta c_{ij} \ , \\ \delta w_i &= -\,\partial_i b + c_{ij}\beta^j \ , & \delta\tilde\beta^\Phi &= c_{ij}\beta^i\beta^j \ . \end{aligned} \tag{2.11}$$

It is easy to see that (2.7), or (2.8), are invariant under the changes (2.11). In general it is not possible to set $w_i = 0$ under such a redefinition except when $w_i = \partial_i X$.

To investigate the consequences of (2.4) we consider the correlation functions sensitive to the additional terms depending on R and $\partial_\mu g$ after restricting to flat space $\gamma_{\mu\nu} \to \delta_{\mu\nu}$ and g^i constant. Thus

$$\big\langle T_{\mu\mu}(x)[\mathcal{O}_i(0)]\big\rangle - \big\langle \Theta(x)[\mathcal{O}_i(0)]\big\rangle = -\,\ell\, w_i\partial^2\delta^2(x) \ , \qquad \Theta = \beta^i[\mathcal{O}_i] \ , \tag{2.12a}$$

$$\big\langle T_{\rho\rho}(x)T_{\mu\nu}(0)\big\rangle - \big\langle \Theta(x)T_{\mu\nu}(0)\big\rangle = -\,\ell\,\beta^\Phi\big(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu\big)\delta^2(x) \ , \tag{2.12b}$$

and hence

$$\big\langle T_{\mu\mu}(x)T_{\nu\nu}(0)\big\rangle = \big\langle \Theta(x)\Theta(0)\big\rangle - \ell\,\tilde\beta^\Phi\partial^2\delta^2(x) \ . \tag{2.13}$$

Furthermore (2.4) and (2.5) give

$$\mathcal{D}\big\langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\big\rangle = 0 \ , \qquad \mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta^i\frac{\partial}{\partial g^i} \ , \tag{2.14a}$$

$$\mathcal{D}\big\langle[\mathcal{O}_i(x)]T_{\mu\nu}(0)\big\rangle + \partial_i\beta^j\big\langle[\mathcal{O}_j(x)]T_{\mu\nu}(0)\big\rangle = \ell\,\partial_i\beta^\Phi\big(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu\big)\delta^2(x) \ , \tag{2.14b}$$

$$\mathcal{D}\big\langle[\mathcal{O}_i(x)][\mathcal{O}_j(0)]\big\rangle + \partial_i\beta^k\big\langle[\mathcal{O}_k(x)][\mathcal{O}_j(0)]\big\rangle + \partial_j\beta^k\big\langle[\mathcal{O}_i(x)][\mathcal{O}_k(0)]\big\rangle = \ell\chi_{ij}\partial^2\delta^2(x) \, . \tag{2.14c}$$

The r.h.s. of (2.14b) is dictated by consistency with (2.14a) and (2.12b) and then combining (2.12a) with (2.14b,c) leads to the essential relation (2.7) once more.

As a consequence of the conservation equation $\partial_\mu T_{\mu\nu} = 0$ we may write

$$\begin{aligned} \big\langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\big\rangle &= \big(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu\big)\big(\partial^2\delta_{\sigma\rho} - \partial_\sigma\partial_\rho\big)\Omega(t) \ , \\ \big\langle[\mathcal{O}_i(x)][\mathcal{O}_j(0)]\big\rangle &= \partial^2\partial^2\Omega_{ij}(t) \ , \qquad t = \tfrac{1}{2}\ln\mu^2x^2 \ . \end{aligned} \tag{2.15}$$

From (2.13) and (2.14a,c), choosing for simplicity $\ell = 4\pi$,

$$\mathcal{D}\Omega' = 0 \ , \quad \Omega''_{ij} + \mathcal{D}\Omega'_{ij} + \partial_i\beta^k\Omega'_{kj} + \partial_j\beta^k\Omega'_{ik} = 2\chi_{ij} \ , \quad \Omega' = -2\tilde\beta^\Phi + \Omega'_{ij}\beta^i\beta^j \ , \tag{2.16}$$

where Ω' denotes the derivative with respect to t (as introduced in (2.15) Ω, Ω_{ij} are arbitrary up to the addition of a constant). In this case (2.9) is crucial for consistency. Using (2.15), (2.16)

$$\begin{aligned} (x^2)^2\big\langle[\mathcal{O}_i(x)][\mathcal{O}_j(0)]\big\rangle &= 8G_{ij}(t) \ , \\ G_{ij} = \tfrac{1}{2}\Omega''_{ij} - \tfrac{1}{2}\Omega'''_{ij} + \tfrac{1}{8}\Omega''''_{ij} &= \chi_{ij} + \mathcal{L}_\beta c_{ij} \ , \quad c_{ij} = -\tfrac{1}{2}\Omega'_{ij} + \tfrac{1}{2}\Omega''_{ij} - \tfrac{1}{8}\Omega'''_{ij} \ , \end{aligned} \tag{2.17}$$

defines G_{ij} to be positive definite and then

$$C = 3\big(\tilde\beta^\Phi + c_{ij}\beta^i\beta^j\big) = -\tfrac{3}{2}\Omega' + \tfrac{3}{2}\Omega'' - \tfrac{3}{8}\Omega''' \tag{2.18}$$

satisfies

$$C' = -\beta^i \partial_i C = -3 G_{ij} \beta^i \beta^j \le 0 \ , \tag{2.19}$$

with equality only if $\beta^i = 0$. Of course this is just Zamolodchikov's c-theorem and is thus shown to be effectively equivalent to the equations (2.9) given the freedom expressed in (2.11). The positive definiteness of G_{ij} is of course crucial for the powerful consequences that have been obtained in applications of the c-theorem. It is easy to show that C as given in (2.18) is exactly the same linear combination of $z^4 \langle T_{zz}(x) T_{zz}(0)\rangle$, $z^2 x^2 \langle T_{zz}(x) T_{z\bar z}(0)\rangle$ and $(x^2)^2 \langle T_{z\bar z}(x) T_{z\bar z}(0)\rangle$ considered by Zamolodchikov. At increasing distance scales $t \to \infty$ and C decreases monotonically until it reaches a fixed point when $\beta^i = 0$. At such a critical point then from (2.15) with $\ell = 4\pi$

$$\big\langle T_{zz}(x) T_{zz}(0) \big\rangle = \frac{6\beta^\Phi}{z^4} \ , \quad \big\langle T_{\mu\mu}(x) T_{\nu\nu}(0) \big\rangle = -4\pi \beta^\Phi \partial^2 \delta^2(x) \ . \tag{2.20}$$

The residual local contribution to $\langle T_{\mu\mu}(x) T_{\nu\nu}(0)\rangle$ may be removed by an appropriate counterterm breaking two dimensional reparameterisation invariance in S. We should stress that although $\langle T_{\mu\nu}(x) T_{\sigma\rho}(0)\rangle$ and $\langle [\mathcal{O}_i(x)][\mathcal{O}_j(0)]\rangle$ have non integrable singularities $\mathrm{O}(x^{-4})$ as $x \to 0$ in general writing expressions for them in the form (2.15), in terms of Ω and Ω_{ij}, ensures that there is a well defined Fourier transform, after integrating the derivatives acting on Ω, Ω_{ij} by parts, giving therefore suitable regularised momentum space amplitudes.

It is also of interest to consider the extension of the consistency relations to field theories defined on a two dimensional manifold with a boundary as would be relevant for the consideration of open strings. In general it is necessary to specify appropriate boundary conditions for the fields but a natural extension of (2.4) to this case is

$$\begin{aligned} \Delta^W_\sigma W = \Delta^\beta_\sigma W &- \frac{1}{\ell} \int dv\, \sigma \big(\tfrac{1}{2} \beta^\Phi R - \tfrac{1}{2} \chi_{ij} \partial_\mu g^i \partial^\mu g^j \big) + \frac{1}{\ell} \int dv\, \partial_\mu \sigma\, w_i \partial^\mu g^i \\ &- \frac{1}{\ell} \int ds\, \sigma \big(\beta^{\hat\Phi} K + n^\mu \omega_i \partial_\mu g^i \big) + \frac{1}{\ell} \int ds\, \partial_\mu \sigma\, n^\mu \epsilon \ . \end{aligned} \tag{2.21}$$

Here s is the arc length along the boundary, n^μ is the unit inward normal to the boundary while K is the extrinsic curvature. Under a local scale transformation

$$\delta ds = -\sigma ds, \qquad \delta n^\mu = \sigma n^\mu \ , \qquad \delta K = \sigma K + n^\mu \partial_\mu \sigma \ , \tag{2.22}$$

Hence (2.6) is modified and in addition to (2.7) we obtain the relation

$$\beta^{\hat\Phi} - \beta^\Phi = \omega_i \beta^i + \beta^i \partial_i \epsilon \ . \tag{2.23}$$

If instead of (2.10)

$$\delta W = \frac{1}{\ell} \int dv \, \big(\tfrac{1}{2} b R - \tfrac{1}{2} c_{ij} \partial_\mu g^i \partial^\mu g^j \big) + \frac{1}{\ell} \int ds \big(\hat b K + n^\mu d_i \partial_\mu g^i \big) \ , \tag{2.24}$$

then, as well as (2.11),

$$\delta\beta^{\hat{\Phi}} = \beta^i \partial_i \hat{b} \ , \qquad \delta\epsilon = \hat{b} - b - d_i\beta^i \ , \qquad \delta\omega_i = \mathcal{L}_\beta d_i \ , \tag{2.25}$$

so that it is possible to set $\epsilon = 0$. When $\partial_\mu g^i = 0$ the requirement of local scale invariance so that $n^\mu\partial_\mu\sigma$ terms are absent on the boundary requires $\epsilon = 0$ and then the result (2.23) shows that at a critical point when also $\beta^i = 0$ the Weyl anomaly is proportional to the Euler density.

3. Four Dimensional Field Theories

The analysis described in the previous section may be extended analogously to four dimensional field theories although the elegant simplicity of Zamolodchikov's c-theorem appears to be lost. We assume a similar framework and suppose now that g^i are the couplings corresponding to a complete set of scalar dimension four operators $\mathcal{O}_i$ in a renormalisable field theory defined with some definite regularisation prescription. For the most part we neglect lower dimension operators which generally require treatment within the context of specific field theories.

As before we allow $g^i(x)$ to be arbitrary functions so that they act as sources for the local operators $\mathcal{O}_i(x)$, defined as in (2.1), and also consider a general curved background metric $\gamma_{\mu\nu}(x)$. The extension of the local renormalisation group equation (2.4) to the present case is then (taking $\ell = 1$).

$$\Delta^W_\sigma W = \Delta^\beta_\sigma W + \int dv\, \sigma\, \mathcal{B}\cdot\mathcal{R} + \int dv\, \partial_\mu\sigma \mathcal{Z}^\mu \ , \qquad dv = d^4x\sqrt{\gamma} \ , \tag{3.1}$$

where $\mathcal{B} = (\beta_a, \beta_b, \beta_c, \chi^e_i, \chi^f_{ij}, \chi^g_{ij}, \chi^a_{ij}, \chi^b_{ijk}, \chi^c_{ijk\ell})$ and

$$\begin{aligned}\mathcal{B}\cdot\mathcal{R} = {}& \beta_a\, F + \beta_b\, G + \tfrac{1}{9}\beta_c\, R^2 \\ & + \tfrac{1}{3}\chi^e_i \partial_\mu g^i\, \partial^\mu R + \tfrac{1}{6}\chi^f_{ij}\partial_\mu g^i \partial^\mu g^j\, R + \tfrac{1}{2}\chi^g_{ij}\partial_\mu g^i \partial_\nu g^j\, G^{\mu\nu} \\ & + \tfrac{1}{2}\chi^a_{ij}\, \nabla^2 g^i \nabla^2 g^j + \tfrac{1}{2}\chi^b_{ijk}\, \partial_\mu g^i \partial^\mu g^j \nabla^2 g^k + \tfrac{1}{4}\chi^c_{ijk\ell}\, \partial_\mu g^i \partial^\mu g^j \partial_\nu g^k \partial^\nu g^\ell \ .\end{aligned} \tag{3.2}$$

Besides the scalar curvature R this also involves

$$\begin{aligned} F &= R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta} R_{\alpha\beta} + \tfrac{1}{3}R^2 \ , \\ G &= R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta} R_{\alpha\beta} + R^2 \ , \qquad G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}\gamma_{\alpha\beta} R \ .\end{aligned} \tag{3.3}$$

F is the square of the conformal Weyl tensor while G is the Euler density. $\mathcal{Z}_\mu$ may also be expanded in the general form

$$\begin{aligned}\mathcal{Z}_\mu = {}& G_{\mu\nu}\, w_i \partial^\nu g^i + \tfrac{1}{3}\partial_\mu(dR) + \tfrac{1}{3}R\, Y_i \partial_\mu g^i \\ & + \partial_\mu\big(U_i \nabla^2 g^i + \tfrac{1}{2}V_{ij}\partial_\nu g^i \partial^\nu g^j\big) + S_{ij}\, \partial_\mu g^i \nabla^2 g^j + \tfrac{1}{2}T_{ijk}\, \partial_\nu g^i \partial^\nu g^j \partial_\mu g^k \ ,\end{aligned} \tag{3.4}$$

up to terms with a vanishing divergence. In (3.2) and (3.4) the various components introduced are scalars or appropriate tensors under redefinition of the couplings, $g^i \to g'^i(g)$, except for χ^b_{ijk}, $\chi^c_{ijk\ell}$ and V_{ij}, T_{ijk} as a consequence of $\nabla^2 g^i$ not transforming as a contravariant vector. As before the essential content of (3.1) is that the response to a Weyl rescaling given by $\gamma^{\mu\nu}\langle T_{\mu\nu}\rangle$ is equal to $\langle\Theta\rangle$, $\Theta = \beta^i[\mathcal{O}_i]$, up to local terms of the form prescribed by (3.2) and (3.4).

The consistency relations follow from the requirement

$$\big[\Delta^W_\sigma - \Delta^\beta_\sigma , \, \Delta^W_{\sigma'} - \Delta^\beta_{\sigma'}\big] W = 0 \ . \tag{3.5}$$

To compute this we use for $\delta\gamma^{\mu\nu} = 2\sigma\,\gamma^{\mu\nu}$

$$\begin{aligned} \delta F = 4\sigma F \ , \quad \delta G = 4\sigma G - 8 G^{\alpha\beta}\nabla_\alpha\nabla_\beta\sigma \ , \quad \delta R = 2\sigma R + 6\nabla^2\sigma \ , \\ \delta G_{\mu\nu} = 2\big(\nabla_\mu\nabla_\nu\sigma - \gamma_{\mu\nu}\nabla^2\sigma\big) \ , \qquad \delta\nabla^2 = 2\sigma\nabla^2 - 2\partial^\mu\sigma\partial_\mu \ , \end{aligned} \tag{3.6}$$

Evaluating (3.5) we find for the term proportional to $\partial_\mu\sigma'\nabla^2\sigma - \nabla^2\sigma'\partial_\mu\sigma$ the requirement

$$\chi^e_i = Y_i - U_i - \partial_i\beta^j U_j - \tfrac{1}{2}(V_{ij} + S_{ij})\beta^j \ , \tag{3.7}$$

and for $\partial_{[\mu}\sigma'\partial_{\nu]}\sigma$

$$w_{[i,j]} = S_{[ij]} - \partial_{[i}\beta^k S_{j]k} - \tfrac{1}{2}\beta^k T_{k[ij]} \ , \tag{3.8}$$

while for $\sigma'\partial_\mu\sigma - \partial_\mu\sigma'\sigma$

$$\begin{aligned} & 8\partial_\mu\beta_b - \chi^g_{ij}\beta^i\partial_\mu g^j + \beta^j\partial_j(w_i\partial_\mu g^i) = 0 \ , \\ & 4\beta_c + \chi^e_i\beta^i - \beta^j\partial_j d = 0 \ , \\ & 4\partial_\mu\beta_c - 2\chi^e_i\partial_\mu g^i + \chi^f_{ij}\beta^i\partial_\mu g^j - \beta^j\partial_j\big(\partial_\mu d + Y_i\partial_\mu g^i\big) = 0 \ , \\ & 2\nabla_\mu\nabla_\nu\big(\chi^e_i\partial^\nu g^i\big) - \nabla_\nu\big(\chi^g_{ij}\partial_\mu g^i\partial^\nu g^j\big) \\ & - \partial_\mu\big((\chi^f_{ij} - \chi^g_{ij})\partial_\nu g^i\partial^\nu g^j - \chi^a_{ij}\beta^i\nabla^2 g^j - \tfrac{1}{2}\chi^b_{ijk}\partial_\nu g^i\partial^\nu g^j\beta^k\big) \\ & - 2\chi^a_{ij}\partial_\mu g^i\nabla^2 g^j - \chi^b_{ijk}\partial_\nu g^i\partial^\nu g^j\partial_\mu g^k - 2\chi^a_{ij}\partial_\mu\beta^i\nabla^2 g^j \\ & - \chi^b_{ijk}\big(\beta^i\partial_\mu g^j\nabla^2 g^k + \partial_\nu g^i\partial^\nu g^j\partial_\mu\beta^k\big) - \chi^c_{ijk\ell}\,\beta^i\partial_\mu g^j\partial^\nu g^k\partial_\nu g^\ell \\ & + \beta^k\partial_k\big(\partial_\mu(U_i\nabla^2 g^i + \tfrac{1}{2}V_{ij}\partial_\nu g^i\partial^\nu g^j) + S_{ij}\partial_\mu g^i\nabla^2 g^j + \tfrac{1}{2}T_{ijk}\partial_\nu g^i\partial^\nu g^j\partial_\mu g^k\big) = 0 \ . \end{aligned} \tag{3.9}$$

This may be decomposed into the separate equations

$$8\partial_i\beta_b - \chi^g_{ij}\,\beta^j = -\mathcal{L}_\beta w_i \ , \tag{3.10a}$$

$$2\chi^e_i + \chi^a_{ij}\,\beta^j = -\mathcal{L}_\beta U_i \ , \tag{3.10b}$$

$$8\beta_c - \chi^a_{ij}\beta^i\beta^j = \mathcal{L}_\beta\big(2d + U_i\beta^i\big) \ , \tag{3.10c}$$

$$4\partial_i\beta_c + (\chi^f_{ij} + \chi^a_{ij})\beta^j = \mathcal{L}_\beta(\partial_i d + Y_i - U_i) \ , \tag{3.10d}$$

$$\chi^g_{ij} + 2\chi^a_{ij} + \Lambda_{ij} = \mathcal{L}_\beta S_{ij} \ , \qquad \Lambda_{ij} = 2\partial_i\beta^k\chi^a_{kj} + \beta^k\chi^b_{kij} \ , \tag{3.10e}$$

$$2\big(\chi^f_{ij} + \chi^a_{ij}\big) + \Lambda_{ij} + \beta^k\big(2\bar\chi^a_{k(ij)} - \bar\chi^a_{ijk}\big) = \mathcal{L}_\beta\big(S_{ij} - \chi^a_{ij} - 2U_{(i,j)} + V_{ij}\big) \ , \tag{3.10f}$$

$$\bar\chi^a_{ijk} = \chi^a_{ij,k} - \chi^b_{k(ij)}$$

$$\chi^b_{ijk} + \chi^g_{k(i,j)} - \tfrac{1}{2}\chi^g_{ij,k} + \partial_k\beta^\ell\chi^b_{ij\ell} + \chi^c_{ijk\ell}\beta^\ell = \tfrac{1}{2}\mathcal{L}_\beta T_{ijk} + \partial_i\partial_j\beta^\ell S_{k\ell} \ , \tag{3.10g}$$

where $\mathcal{L}_\beta$ is again the Lie derivative defined by the vector field β^i. These equations are not independent. Using $\bar\chi^a_{kji}\beta^j\beta^k + \Lambda_{ij}\beta^j = \partial_i(\chi^a_{jk}\beta^j\beta^k)$ it is clear that (3.10f), (3.10b) and (3.7) are sufficient for the compatibility of (3.10d) with (3.10c). Furthermore contracting (3.10g) with β^i, antisymmetrising on j,k and combining with (3.10e) gives

$$\partial_{[j}\big(\chi^g_{i]k}\beta^k\big) = \mathcal{L}_\beta\big(S_{[ij]} - \partial_{[i}\beta^k S_{j]k} - \tfrac{1}{2}\beta^k T_{k[ij]}\big) \ ,$$

so that (3.8) is necessary for the integrability of (3.10a).

In the four dimensional case the potential arbitrariness in W is given by, with a similar notation to (3.2),

$$\delta W = \int dv\, B\cdot\mathcal{R} \ , \qquad B = (a,b,c,e_i,f_{ij},g_{ij},a_{ij},b_{ijk},c_{ijk\ell}) \ . \tag{3.11}$$

This gives

$$\begin{aligned}
\delta(\beta_a,\beta_b,\beta_c,\chi^e_i,\chi^f_{ij},\chi^g_{ij},\chi^a_{ij}) &= \mathcal{L}_\beta(a,b,c,e_i,f_{ij},g_{ij},a_{ij}) \ ,\\
\delta\chi^b_{ijk} &= \mathcal{L}_\beta b_{ijk} + 2\partial_i\partial_j\beta^\ell a_{\ell k} \ ,\\
\delta\chi^c_{ijk\ell} &= \mathcal{L}_\beta c_{ijk\ell} + \partial_i\partial_j\beta^m b_{k\ell m} + \partial_k\partial_\ell\beta^m b_{ijm} \ ,\\
\delta w_i &= -8\partial_i b + g_{ij}\beta^j \ , \qquad \delta d = 4c + e_i\beta^i \ ,\\
\delta U_i &= -2e_i - a_{ij}\beta^j \ , \qquad \delta Y_i = -2e_i - \partial_i(e_j\beta^j) + f_{ij}\beta^j \ ,\\
\delta V_{ij} &= -4e_{(i,j)} + 2f_{ij} - g_{ij} - b_{ijk}\beta^k \ ,\\
\delta S_{ij} &= g_{ij} + 2a_{ij} + 2\partial_i\beta^k a_{kj} + b_{kij}\beta^k \ ,\\
\delta T_{ijk} &= 2g_{k(i,j)} - g_{ij,k} + 2b_{ijk} + 2\partial_k\beta^\ell b_{ij\ell} + 2c_{ijk\ell}\beta^\ell \ .
\end{aligned} \tag{3.12}$$

It is straightforward to check that the consistency equations (3.7), (3.8) and (3.10) are invariant under this arbitrariness. Clearly this freedom may be partially fixed by setting d, Y_i or U_i, V_{ij}, $S_{(ij)}$, T_{ijk} to zero. The results for $\delta\beta_b$, δw_i and the equation (3.10a) are essentially identical to (2.11) and (2.7) which led to the c-theorem for two dimensional theories. In this case we may also obtain as in (2.9)

$$\beta^i\partial_i\tilde\beta_b = \tfrac{1}{8}\chi^g_{ij}\beta^i\beta^j \ , \qquad \tilde\beta_b = \beta_b + \tfrac{1}{8}w_i\beta^i \ , \tag{3.13}$$

which might bear the same relation to a four dimensional version of the c-theorem as (2.9) to (2.19).

In the same fashion as previously we may explore the consequences of the basic equation (3.1), with (3.2) and (3.4), for two point correlation functions after restricting to flat space, $\gamma_{\mu\nu} = \delta_{\mu\nu}$, and g^i constant when we suppose also that $\langle[\mathcal{O}_i]\rangle = 0$. Analogous to (2.12) we have

$$\big\langle T_{\mu\mu}(x)[\mathcal{O}_i(0)]\big\rangle - \big\langle\Theta(x)[\mathcal{O}_i(0)]\big\rangle = -U_i\partial^2\partial^2\delta^4(x) \ , \quad \Theta = \beta^i[\mathcal{O}_i] \ , \tag{3.14a}$$

$$\big\langle T_{\rho\rho}(x)T_{\mu\nu}(0)\big\rangle - \big\langle\Theta(x)T_{\mu\nu}(0)\big\rangle = -\tfrac{2}{3}d\,S_{\mu\nu}\partial^2\delta^4(x) \ , \ S_{\mu\nu} = \partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu \ , \tag{3.14b}$$

and hence now

$$\big\langle T_{\mu\mu}(x)T_{\nu\nu}(0)\big\rangle = \big\langle \Theta(x)\Theta(0)\big\rangle - 2\tilde{d}\,\partial^2\partial^2\delta^4(x) \ , \qquad \tilde{d} = d + \tfrac{1}{2}U_i\beta^i \ . \tag{3.15}$$

If

$$\big\langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\big\rangle = \tfrac{1}{3}S_{\mu\nu}S_{\sigma\rho}\Omega_0(x) + \big(S_{\mu(\sigma}S_{\rho)\nu} - \tfrac{1}{3}S_{\mu\nu}S_{\sigma\rho}\big)\Omega_2(x) \ , \tag{3.16}$$

then the renormalisation group equation from (3.1), (3.2) and (2.5) becomes

$$\mathcal{D}\Omega_0(x) = -\tfrac{8}{3}\beta_c\delta^4(x) \ , \qquad \mathcal{D}\Omega_2(x) = -4\beta_a\delta^4(x) \ , \tag{3.17}$$

as well as

$$\mathcal{D}\big\langle[\mathcal{O}_i(x)]T_{\mu\nu}(0)\big\rangle + \partial_i\beta^j\big\langle[\mathcal{O}_j(x)]T_{\mu\nu}(0)\big\rangle = \tfrac{2}{3}\chi^e_i S_{\mu\nu}\partial^2\delta^4(x) \ , \tag{3.18a}$$

$$\mathcal{D}\big\langle[\mathcal{O}_i(x)][\mathcal{O}_j(0)]\big\rangle + \partial_i\beta^k\big\langle[\mathcal{O}_k(x)][\mathcal{O}_j(0)]\big\rangle + \partial_j\beta^k\big\langle[\mathcal{O}_i(x)][\mathcal{O}_k(0)]\big\rangle = -\chi^a_{ij}\partial^2\partial^2\delta^4(x) \ . \tag{3.18b}$$

It is easy to check that the consistency of (3.17) and (3.18a,b) with (3.14a,b) is equivalent to (3.10b,c). Other relations in (3.10) require the consideration of correlation functions involving three or more operators.

As an illustration of the consequences of (3.17) we obtain here a result due to Shore [19] by following analogous steps in four dimensions to the derivation of the original c-theorem. For flat space letting

$$G(t) = 2\pi^4(x^2)^4\Big(\big\langle T_{\mu\nu}(x)T_{\mu\nu}(0)\big\rangle - \tfrac{1}{3}\big\langle T_{\mu\mu}(x)T_{\nu\nu}(0)\big\rangle\Big) = 10\pi^4(x^2)^4\partial^2\partial^2\Omega_2(x) \ , \tag{3.19}$$

for $t = \frac{1}{2}\ln\mu^2x^2$ again, defines $G(t)$ as positive definite. If

$$\Omega_2(x) = \frac{1}{(x^2)^2}\,f(t) + c\,\delta^4(x) = \partial^2\Big(\frac{1}{x^2}\,h(t)\Big) \ , \qquad f = -2h' + h'' \ ,$$

where the expression in terms of h ensures a well defined Fourier transform for the momentum space amplitude, then (3.17) and (3.19) imply

$$h'\big|_{\beta^i=0} = \frac{\beta_a}{\pi^2} \ , \quad G = 10\pi^4\big(192f - 224f' + 92f'' - 16f''' + f''''\big) \ . \tag{3.20}$$

Hence

$$C_F = 10\pi^4\big(96f - 64f' + 14f'' - f'''\big) \tag{3.21}$$

satisfies

$$C_F' = -\beta^i\frac{\partial}{\partial g^i}C_F = 2C_F - G \ , \qquad C_F\big|_{\beta^i=0} = -30\times 64\pi^2\,\beta_a \ , \tag{3.22}$$

as obtained by Shore. Clearly in this case the renormalisation flow of C_F is not monotonic. For a free theory with n_V vectors, n_F Dirac fermions and n_S scalars [24]

$$C_F = 12n_V + 6n_F + n_S \ . \tag{3.23}$$

In the above we have neglected operators of dimension lower than four. In general to remedy this requires a separate discussion in each particular theory but here we consider the presence of a set of dimension two operators $\mathcal{O}^m_a(x)$ with couplings, or mass^2 terms, $m^a(x)$ as occurs in scalar field theories corresponding to operators ϕ^2. In this case (3.1) may in general be written in the form

$$\begin{aligned} \hat\Delta^W_\sigma W &= \big(\hat\Delta^\beta_\sigma + \Delta^m_\sigma\big)W + \int dv\,\sigma\,\hat{\mathcal{B}}\cdot\mathcal{R} + \int dv\,\partial_\mu\sigma\,\hat{\mathcal{Z}}^\mu \ , \\ \hat{\mathcal{B}}\cdot\mathcal{R} &= \mathcal{B}\cdot\mathcal{R} + \tfrac12\,p_{ab}\hat m^a\hat m^b + \hat m^a\big(\tfrac13 q_a R + r_{ai}\nabla^2 g^i + \tfrac12 s_{aij}\partial_\mu g^i\partial^\mu g^j\big) \ , \\ \hat{\mathcal{Z}}_\mu &= \mathcal{Z}_\mu + \hat m^a j_{ai}\partial_\mu g^i + \partial_\mu(\hat m^a k_a) \ , \qquad \hat m^a = m^a - \tfrac16\,\tau^a R \ , \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} \hat\Delta^W_\sigma &= \Delta^W_\sigma + 2\int dv\,\sigma\,m^a\frac{\delta}{\delta m^a} \ , \qquad \hat\Delta^\beta_\sigma = \Delta^\beta_\sigma - \int dv\,\sigma\,m^a\gamma_a{}^b\frac{\delta}{\delta m^b} \ , \\ \Delta^m_\sigma &= -\int dv\,\sigma\,\big(\tfrac13\beta^a_\eta\,R + \delta^a_i\nabla^2 g^i + \tfrac12\epsilon^a_{ij}\partial_\mu g^i\partial^\mu g^j\big)\frac{\delta}{\delta m^a} \\ &\quad + \int dv\,\partial_\mu\sigma\,\theta^a_i\partial^\mu g^i\frac{\delta}{\delta m^a} - \int dv\,\nabla^2\sigma\,\tau^a\frac{\delta}{\delta m^a} \ , \end{aligned} \tag{3.25}$$

with $\gamma_a{}^b(g)$ the anomalous dimension matrix for the operators coupled to m^a and similarly the various coefficients appearing in (3.24), (3.25) depend on the dimensionless couplings g^i. For scalar field theories $\hat\Delta^W_\sigma S$ is non zero, unless the kinetic term has the conformally invariant form $\frac12(\partial\phi)^2 + \frac1{12}R\phi^2$, but this may be compensated at zero loop order in (3.24) by the appropriate choice of τ^a. In (3.24) defining the additional contributions in terms of $\hat m^a$, rather than just m^a, ensures some simplification subsequently. Since $[\Delta^m_\sigma,\Delta^m_{\sigma'}]=0$ the consistency conditions flowing from (3.24), (3.25) can be decomposed as

$$\big[\hat\Delta^W_\sigma - \hat\Delta^\beta_\sigma,\,\Delta^m_{\sigma'}\big] - (\sigma\leftrightarrow\sigma') = 0 \ , \tag{3.26a}$$

$$\big(\hat\Delta^W_\sigma - \hat\Delta^\beta_\sigma - \Delta^m_\sigma\big)\Big(\int dv\,\sigma'\,\hat{\mathcal{B}}\cdot\mathcal{R} + \int dv\,\partial_\mu\sigma'\,\hat{\mathcal{Z}}^\mu\Big) - (\sigma\leftrightarrow\sigma') = 0 \ . \tag{3.26b}$$

From (3.26a) we obtain

$$2\beta^a_\eta - \delta^a_i\beta^i = -\hat{\mathcal{L}}_\beta\tau^a = -\mathcal{L}_\beta\tau^a - \tau^b\gamma_b{}^a \ , \tag{3.27a}$$

$$2\delta^a_i + 2\partial_i\beta^j\delta^a_j + \epsilon^a_{ij}\beta^j = -\hat{\mathcal{L}}_\beta\theta^a_i \ , \tag{3.27b}$$

where $\hat{\mathcal{L}}_\beta$ is the extension of the Lie derivative defined by β^i to include also transformation by the matrix $\gamma_a{}^b$. Allowing for a local change in the couplings m^a effected by

$$\delta W = \int dv\,\big(\tfrac13 f^a R + d^a_i\nabla^2 g^i + \tfrac12 e^a_{ij}\partial_\mu g^i\partial^\mu g^i\big)\frac{\delta}{\delta m^a}W \ , \tag{3.28}$$

then

$$\begin{aligned} &\delta\beta^\eta_a = \hat{\mathcal{L}}_\beta f^a \ , \quad \delta\delta^a_i = \hat{\mathcal{L}}_\beta d^a_i \ , \quad \delta\epsilon^a_{ij} = \hat{\mathcal{L}}_\beta e^a_{ij} + 2\partial_i\partial_j\beta^k d^a_k \ , \\ &\delta\tau^a = -2f^a + d^a_i\beta^i \ , \qquad \delta\theta^a_i = -2d^a_i - 2\partial_i\beta^j d^a_j - e^a_{ij}\beta^j \ , \end{aligned} \tag{3.29}$$

so that it is possible to set τ^a and θ_i^a to zero. β_η^a is related to the so called improvement term in the energy momentum tensor on flat space for scalar field theories.

The extra terms arising in (3.26b) may be calculated by using (3.27a) to obtain

$$
\begin{aligned}
(\hat{\Delta}_\sigma^W - \hat{\Delta}_\sigma^\beta - \Delta_\sigma^m)\hat{m}^a = {}& 2\sigma\,\hat{m}^a + \sigma\,\hat{m}^b\gamma_b{}^a \\
& + \sigma\big(\tfrac{1}{6}\delta_i^a\beta^i R + \delta_i^a\nabla^2 g^i + \tfrac{1}{2}\epsilon_{ij}^a\partial_\mu g^i\partial^\mu g^j\big) - \partial_\mu\sigma\,\theta_i^a\partial^\mu g^i \ .
\end{aligned} \tag{3.30}
$$

The part proportional to $\hat{m}$ gives

$$
2q_a - r_{ai}\beta^i = \hat{\mathcal{L}}_\beta k_a = \mathcal{L}_\beta k_a - \gamma_a{}^b k_b \ , \tag{3.31a}
$$

$$
2r_{ai} + 2\partial_i\beta^j r_{aj} + s_{aij}\beta^j + p_{ab}\theta_i^b = \hat{\mathcal{L}}_\beta j_{ai} \ . \tag{3.31b}
$$

In addition (3.26b) leads to additional terms in all the previous consistency equations except (3.10a). Thus we find

$$
\begin{aligned}
& 2\chi_i^e = Y_i - U_i - \partial_i\beta^j U_j - \tfrac{1}{2}(V_{ij} + S_{ij})\beta^j - \theta_i^a k_a \\
& \qquad = -\chi_{ij}^a\,\beta^j + \delta_i^a k_a - \mathcal{L}_\beta U_i \ , \\
& 8\beta_c - \chi_{ij}^a\beta^i\beta^j + 2\delta_i^a k_a = \mathcal{L}_\beta\big(2d + U_i\beta^i\big) \ , \\
& \partial_i\big(8\beta_c + \delta_j^a k_a\beta^j\big) + 2(\chi_{ij}^f + \chi_{ij}^a)\beta^j - 2\delta_i^a k_a + 2\theta_i^a q_a + j_{ai}\delta_j^a\beta^j \\
& \qquad = 2\mathcal{L}_\beta(\partial_i d + Y_i - U_i) \ , \\
& \chi_{ij}^g + 2\chi_{ij}^a + \Lambda_{ij} + \rho_{ij} = \mathcal{L}_\beta S_{ij} \ , \qquad \rho_{ij} = \theta_i^a r_{aj} + \delta_j^a j_{ai} \ , \\
& 2\big(\chi_{ij}^f + \chi_{ij}^a\big) + \Lambda_{ij} + \rho_{ij} + \beta^k\big(2\bar{\chi}_{k(ij)}^a - \bar{\chi}_{ijk}^a\big) - 2\partial_{(i}(\delta_{j)}^a k_a) + \epsilon_{ij}^a k_a \\
& \qquad = \mathcal{L}_\beta\big(S_{ij} - \chi_{ij}^a - 2U_{(i,j)} + V_{ij}\big) \ .
\end{aligned} \tag{3.32}
$$

A non trivial check is that, as before, the equation for $\partial_i\beta_c$ is implied by the explicit equation for β_c. The modifications of the earlier relations are a reflection of operator mixing involving derivatives of the scalar operators $\mathcal{O}_a^m$. As an illustration of this we may note that from (3.24) now, on flat space and for constant couplings,

$$
\Theta = \beta^i[\mathcal{O}_i] - \tau^a\partial^2[\mathcal{O}_a^m] + \mathrm{O}(m) \ , \tag{3.33}
$$

where the finite local operator $[\mathcal{O}_a^m(x)]$ is defined by functional differentiation with respect to $m^a(x)$, and the operators have anomalous dimensions again determined by (3.24) so that

$$
\begin{aligned}
\mathcal{D}\begin{pmatrix} T_{\mu\nu} \\ [\mathcal{O}_a^m] \end{pmatrix} &= \begin{pmatrix} 0 & -\tfrac{1}{3}\beta_\eta^b\,S_{\mu\nu} \\ 0 & \gamma_a{}^b \end{pmatrix}\begin{pmatrix} T_{\mu\nu} \\ [\mathcal{O}_b^m] \end{pmatrix} \ , \\
\mathcal{D}\begin{pmatrix} [\mathcal{O}_i] \\ [\mathcal{O}_a^m] \end{pmatrix} &= \begin{pmatrix} -\partial_i\beta^j & \delta_i^b\,\partial^2 \\ 0 & \gamma_a{}^b \end{pmatrix}\begin{pmatrix} [\mathcal{O}_j] \\ [\mathcal{O}_b^m] \end{pmatrix} \ .
\end{aligned} \tag{3.34}
$$

The compatibility of the operator equation $T_{\mu\mu} = \Theta$ with (3.33), (3.34) requires (3.27a). Instead of (3.14a,b) we now have, with Θ given by (3.33),

$$
\big\langle T_{\mu\mu}(x)[\mathcal{O}_i(0)]\big\rangle - \big\langle\Theta(x)[\mathcal{O}_i(0)]\big\rangle = -\,U_i\,\partial^2\partial^2\delta^4(x) \ , \tag{3.35a}
$$

$$
\big\langle T_{\mu\mu}(x)[\mathcal{O}_a^m(0)]\big\rangle - \big\langle\Theta(x)[\mathcal{O}_a^m(0)]\big\rangle = -\,k_a\,\partial^2\delta^4(x) \ , \tag{3.35b}
$$

$$
\big\langle T_{\rho\rho}(x)T_{\mu\nu}(0)\big\rangle - \big\langle\Theta(x)T_{\mu\nu}(0)\big\rangle = -\,\tfrac{1}{3}(2d - \tau^a k_a)\,S_{\mu\nu}\partial^2\delta^4(x) \ , \tag{3.35c}
$$

and in addition the renormalisation group equations for the two point correlation functions from (3.24) become instead of (3.17)

$$\mathcal{D}\Omega_0(x) = -\tfrac{1}{3}\big(8\beta_c + p_{ab}\tau^a\tau^b - 4q_a\tau^a\big) \ , \tag{3.36}$$

and replacing (3.18a,b)

$$\begin{aligned}
\mathcal{D}\big\langle[\mathcal{O}_i(x)]T_{\mu\nu}(0)\big\rangle + \ldots &= \tfrac{1}{3}(2\chi^e_i + \tau^a r_{ai})\, S_{\mu\nu}\partial^2\delta^4(x) \ , && (3.37a)\\
\mathcal{D}\big\langle[\mathcal{O}^m_a(x)]T_{\mu\nu}(0)\big\rangle + \ldots &= -\tfrac{1}{3}(2q_a - p_{ab}\tau^b)\, S_{\mu\nu}\delta^4(x) \ , && (3.37b)\\
\mathcal{D}\big\langle[\mathcal{O}_i(x)][\mathcal{O}_j(0)]\big\rangle + \ldots &= -\chi^a_{ij}\,\partial^2\partial^2\delta^4(x) \ , && (3.37c)\\
\mathcal{D}\big\langle[\mathcal{O}^m_a(x)][\mathcal{O}_j(0)]\big\rangle + \ldots &= -r_{aj}\,\partial^2\delta^4(x) \ , && (3.37d)\\
\mathcal{D}\big\langle[\mathcal{O}^m_a(x)][\mathcal{O}^m_b(0)]\big\rangle + \ldots &= -p_{ab}\,\delta^4(x) \ , && (3.37e)
\end{aligned}$$

neglecting anomalous dimension terms, as required by (3.34), on the l.h.s. of each of the above equations. By applying $\mathcal{D}$ to both sides of (3.35a,b,c) and using (3.33) and (3.36), (3.37a,b,c,d) we may rederive the consistency relations for β_c, χ^e_i in (3.32) and also (3.31a). Presumably the remaining results could be obtained similarly by looking at higher point functions.

For theories containing scalar or fermion fields ϕ there are also spin 1 composite operators of dimension 3. To allow for arbitrary such operators in this framework we suppose that the maximal non anomalous symmetry group G of the kinetic term ($O(n)$ for n real scalar fields) is extended to a symmetry of the interacting theory by requiring that under an infinitesimal transformation $\delta\phi = -\omega\phi$, where ω is an element of the Lie algebra of G, the couplings also transform according to the appropriate representation $\delta g^i = -(\omega^g g)^i$ to ensure S is invariant. For simplicity we neglect other lower dimension operators, the analysis is straightforwardly extended to include these. This symmetry becomes a local gauge symmetry when external background gauge fields A_μ, also belonging to the Lie algebra of G with $\delta A_\mu = D_\mu\omega = \partial_\mu\omega + [A_\mu,\omega]$, are introduced so that $\partial_\mu\phi \to \partial_\mu\phi + A_\mu\phi$. For generators of the quantum gauge group it is of course necessary that $(\omega^g g)^i = 0$. The arbitrary vector fields $A_\mu(x)$ are then treated as additional couplings, like $g^i(x)$, whose variation define the currents as local composite operators $\langle J^\mu\rangle = -\delta W/\delta A_\mu$. Assuming the regularisation procedure preserves local gauge invariance then

$$\int dv\,\Big(D_\mu\omega\cdot\frac{\delta}{\delta A_\mu} - (\omega^g g)^i\frac{\delta}{\delta g^i}\Big)W = 0 \ , \tag{3.38}$$

with $\cdot$ denoting the invariant scalar product on the Lie algebra of G. This is equivalent to the partial conservation equation for $D_\mu\langle J^\mu\rangle$ expressing an operator identity for J^μ subject to the equations of motion. The essential equation (3.1) may then be extended to allow for dimension 3 vector operators by replacing

$$\partial_\mu g^i \to D_\mu g^i = \partial_\mu g^i + (A^g_\mu g)^i \ , \tag{3.39}$$

and also taking

$$
\begin{gathered}
\Delta^\beta_\sigma \to \Delta^\beta_\sigma + \Delta^A_\sigma \ , \qquad \Delta^A_\sigma = \int dv \, \Big(\sigma \, D_\mu g^i \, \rho_i \cdot \frac{\delta}{\delta A_\mu} - \partial_\mu \sigma \, S \cdot \frac{\delta}{\delta A_\mu} \Big) \ , \\
\mathcal{B} \cdot \mathcal{R} \to \mathcal{B} \cdot \mathcal{R} + \tfrac{1}{4} F_{\mu\nu} \cdot \kappa \cdot F^{\mu\nu} + \tfrac{1}{2} F^{\mu\nu} \cdot \zeta_{ij} \, D_\mu g^i D_\nu g^j \ , \quad \mathcal{Z}^\mu \to \mathcal{Z}^\mu + F^{\mu\nu} \cdot \eta_i \, D_\nu g^i \ ,
\end{gathered} \tag{3.40}
$$

for ρ_i, ζ_{ij}, η_i, S belonging to the Lie algebra of G and, like κ which defines an invariant product on the Lie algebra, depending on g^i. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the usual field strength. Alternatively we may define $\beta^A_\mu = \rho_i D_\mu g^i$ as an additional β function, this has been shown to be necessary by explicit calculation at one loop. The additional contributions in $\mathcal{B} \cdot \mathcal{R}$ involving κ, ζ_{ij} produce inhomogeneous terms in the renormalisation group equations involving correlation functions for $\langle J^\mu J^\nu \rangle$ and $\langle J^\mu \mathcal{O}_i \mathcal{O}_j \rangle$ after setting A_μ and $\partial_\mu g^i$ to zero. By using (3.38) the term involving S in $\mathcal{Z}^\mu$ may be eliminated at the expense of letting

$$
\beta^i \to B^i = \beta^i - (S^g g)^i \ , \qquad \rho_i \to P_i = \rho_i + \partial_i S \ . \tag{3.41}
$$

In dimensional regularisation with minimal subtraction there is a well defined prescription for determining S, although to the low orders calculated $S = 0$ [14]. This modification cancels the potential arbitrariness in the β functions as a consequence of the freedom to make G transformations on the couplings so that Θ is unambiguously defined. When $D_\mu g^i = 0$

$$
\Theta = B^i [\mathcal{O}_i] - \beta_a \, F - \beta_b \, G - \tfrac{1}{9} \beta_c \, R^2 + \tfrac{1}{3} d \, \nabla^2 R - \tfrac{1}{4} \, F_{\mu\nu} \cdot \kappa \cdot F^{\mu\nu} \ . \tag{3.42}
$$

With the changes necessitated by (3.40) there are modifications to the consistency relations. These may be calculated by using

$$
\begin{gathered}
\big(\Delta^\beta_\sigma + \Delta^A_\sigma \big) D_\mu g^i = \partial_\mu \sigma \, B^i + \sigma \big(\gamma^i{}_j D_\mu g^j + (S^g D_\mu g)^i \big) \ , \quad \gamma^i{}_j = \partial_j B^i + (P^g_j g)^i \ , \\
\big(\Delta^\beta_\sigma + \Delta^A_\sigma \big) D^2 g^i = D^\mu \big(\partial_\mu \sigma \, B^i + \sigma \, \gamma^i{}_j D_\mu g^j \big) + \sigma \, (P^g_j D_\mu g)^i D^\mu g^j + \sigma \, (S^g D^2 g)^i \ , \\
\big(\Delta^\beta_\sigma + \Delta^A_\sigma \big) F_{\mu\nu} = 2 \, D_{[\mu} (\sigma P_i) D_{\nu]} g^i + \sigma \, (F^g_{\mu\nu} g)^i P_i - \sigma \, [F_{\mu\nu}, S] \ .
\end{gathered} \tag{3.43}
$$

Hence

$$
\big[\Delta^\beta_\sigma + \Delta^A_\sigma , \, \Delta^\beta_{\sigma'} + \Delta^A_{\sigma'} \big] = - \int dv \, \big(\sigma \partial_\mu \sigma' - \sigma' \partial_\mu \sigma \big) \, B^i \, P_i \cdot \frac{\delta}{\delta A_\mu} \ ,
$$

so that it is necessary that

$$
B^i P_i = 0 \ . \tag{3.44}
$$

By computing the action of $\Delta^\beta_\sigma + \Delta^A_\sigma$ on the $F_{\mu\nu}$ dependent terms we obtain

$$
\begin{gathered}
\eta_i B^i = 0 \ , \\
t_a \cdot \kappa \cdot P_i + t_a \cdot \zeta_{ji} B^j = t_a \cdot \tilde{\mathcal{L}}_B \eta_i + (t^g_a g)^j P_j \cdot \eta_i \ , \quad \tilde{\mathcal{L}}_B \eta_i = B^j \partial_j \eta_i + \eta_j \gamma^j{}_i \ ,
\end{gathered} \tag{3.45}
$$

where t_a form a basis for the Lie algebra of G and $\tilde{\mathcal{L}}_B$ is a modified Lie derivative. In eqs. (3.10a,b,c,d) the only necessary changes are that $\beta^i \to B^i$ and $\mathcal{L}_\beta \to \tilde{\mathcal{L}}_B$, which is defined by analogy to (3.45). From (3.40) by considering local changes in A_μ it is easy to see that ρ_i, S are arbitrary up to

$$\delta\rho_i = \tilde{\mathcal{L}}_B p_i \ , \qquad \delta S = -B^i p_i \ . \tag{3.46}$$

4. Non Linear σ Models

In order to demonstrate the significance of the Weyl scaling consistency relations in a specific context we here apply the framework described in section 2 to general bosonic renormalisable σ models in two dimensions. In this case it is necessary to take account of lower dimension operators and also of the various symmetries preserved by the quantum field theory. For such σ models the fields $\phi^i(x)$ are coordinates on a target manifold $\mathcal{M}$, of dimension D, and the essential classical action, with x belonging to a general two dimensional space with metric $\gamma_{\mu\nu}$, is

$$S = \int dv \left(\tfrac{1}{2}\, G_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + \tfrac{1}{2}\, i\epsilon^{\mu\nu}B_{ij}\partial_\mu\phi^i\partial_\nu\phi^j + \tfrac{1}{2}\,\Phi R + V_{\mu i}\partial^\mu\phi^i + T\right) . \tag{4.1}$$

$\epsilon^{\mu\nu}$ is the two dimensional antisymmetric symbol, $\epsilon^{12} = 1/\sqrt{\gamma}$, $\nabla_\sigma\epsilon^{\mu\nu} = 0$. G_{ij} defines a metric on $\mathcal{M}$ and in a string context G_{ij}, Φ correspond to the graviton, dilaton while $B_{ij} = -B_{ji}$ is the antisymmetric tensor field and T represents the tachyon of the closed bosonic string. $V_{\mu i}$ is introduced since the presence of a coupling for operators involving a single $\partial_\mu\phi$, having spin 1, is necessary for a consistent quantum field theory subsequently. These real couplings, denoted collectively by

$$\lambda = (G_{ij},\, B_{ij},\, \Phi,\, V_{\mu i},\, T) \ , \qquad g = (G_{ij},\, B_{ij}) \ , \tag{4.2}$$

are assumed to have an arbitrary dependence on x, $\lambda(\phi, x)$, in addition to being appropriate tensor fields on $\mathcal{M}$, so that they also act as sources for the local composite operators of dimension 2 or less, with spin 0 or 1, in the quantum σ model defined by the action in (4.1), $g(\phi, x)$ are the essential renormalisable couplings in the non linear σ model corresponding to the dimensionless couplings g^i of section 2.

The action S prescribed by (4.1) enjoys gauge invariance under the symmetries

$$\begin{aligned} \delta\lambda = \lambda_{w,F_\mu} &= \big(0\,,(\mathrm{d}w)_{ij},\, 0,\, \partial_i F_\mu - i\epsilon_\mu{}^\nu\partial'_\nu w_i,\, \nabla'^\mu F_\mu\big) \ , \\ g_w &= \big(0,\, (\mathrm{d}w)_{ij}\big) \ , \qquad (\mathrm{d}w)_{ij} = \partial_i w_j - \partial_j w_i \ , \end{aligned} \tag{4.3}$$

for arbitrary $w_i(\phi, x)$ and $F_\mu(\phi, x)$. ∂'_μ denotes the derivative with respect to x at constant ϕ. It is useful to define invariants under (4.3)

$$\begin{aligned} H_{ijk} &= \tfrac{1}{2}(\mathrm{d}B)_{ijk} = \tfrac{1}{2}\big(\partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}\big) \ , \\ A_{\mu ij} &= (\mathrm{d}V_\mu)_{ij} + i\epsilon_\mu{}^\nu\partial'_\nu B_{ij} \ , \qquad T'_i = \partial_i T - \nabla'^\mu V_{\mu i} \ , \end{aligned} \tag{4.4}$$

where

$$(\mathrm{d}A_\mu)_{ijk} = 2i\epsilon_\mu{}^\nu\partial'_\nu H_{ijk} \ , \qquad \nabla'^\mu A_{\mu ij} = -(\mathrm{d}T')_{ij} \ . \tag{4.5}$$

$S(\phi,\lambda)$ is also invariant under complex conjugation in association with the operation, denoted by a bar, on the couplings determined by $\lambda \to \bar\lambda$ where $\bar B_{ij} = -B_{ij}$ with otherwise the remaining couplings left unchanged, thus $\bar g = (G_{ij},\, -B_{ij})$. These symmetries are all assumed to be manifestly preserved by the process of regularisation, without therefore any anomalies, and hence are also symmetries of the bare action S_0 defining a finite quantum field theory as in (2.2). Since (4.1) involves $\epsilon_{\mu\nu}$ it is potentially dangerous using dimensional regularisation but appropriate prescriptions preserving the symmetry (4.3) have proved possible [13].

In addition it is essential to maintain invariance under diffeomorphisms on $\mathcal{M}$ when

$$
\begin{aligned}
&\delta\phi^i = -\,v^i\ , \qquad \delta\lambda = \lambda_v^D\ , \qquad g_v^D = \big(\mathcal{L}_v G_{ij},\, 2H_{ijk}v^k\big)\ ,\\
&\lambda_v^D = \big(\mathcal{L}_v G_{ij},\, 2H_{ijk}v^k,\, \mathcal{L}_v\Phi,\, -A_{\mu ij}v^j,\, T'_i v^i\big)\ ,
\end{aligned}
\tag{4.6}
$$

for $\mathcal{L}_v$ the Lie derivative defined by the $v^i(\phi)$ acting on tensor fields on $\mathcal{M}$. λ_v differs from $\mathcal{L}_v\lambda$ by symmetry transformations of the form (4.3), specifically

$$
\begin{aligned}
\mathcal{L}_v B_{ij} &= 2H_{ijk}v^k - ({\rm d}\tilde u)_{ij}\ , \qquad \tilde u_i = B_{ij}v^j\ ,\\
\mathcal{L}_v V_{\mu i} &= -\,A_{\mu ij}v^j + i\epsilon_\mu{}^\nu \partial'_\nu \tilde u_i + \partial_i(V_{\mu j}v^j) - i\epsilon_\mu{}^\nu B_{ij}\partial'_\nu v^j\ ,\\
\mathcal{L}_v T &= T'_i v^i + \nabla'^\mu (V_{\mu i}v^i) - V^\mu{}_i \partial'_\mu v^i\ ,
\end{aligned}
\tag{4.7}
$$

allowing now for general $v^i(\phi,x)$.

The requirement of preserving the symmetry of S_0 under diffeomorphisms is expressed as

$$
\begin{aligned}
&\Delta_v^D S_0 = -\,\Delta_v'^D S_0\ ,\\
&\Delta_v^D = \int dv\, \Big(-v^i\frac{\delta}{\delta\phi^i} + \lambda_v^D\cdot\frac{\delta}{\delta\lambda}\Big)\ , \qquad \Delta_v'^D = \int dv\, \big(G_{ij}\partial'_\mu v^j\big)\cdot\frac{\delta}{\delta V_{\mu i}}\ ,
\end{aligned}
\tag{4.8}
$$

while invariance under (4.3) is formally specified by

$$
\Delta_w^g S_0 = 0\ , \qquad \Delta_w^g = \int dv\, \Big(({\rm d}w)_{ij}\cdot\frac{\delta}{\delta B_{ij}} - i\epsilon_\mu{}^\nu\partial'_\nu w_i\cdot\frac{\delta}{\delta V_{\mu i}}\Big)\ , \tag{4.9a}
$$

$$
\Delta_{F_\mu}^g S_0 = 0\ , \qquad \Delta_{F_\mu}^g = \int dv\, \Big(\partial_i F_\mu\cdot\frac{\delta}{\delta V_{\mu i}} + \nabla'^\mu F_\mu\cdot\frac{\delta}{\delta T}\Big)\ . \tag{4.9b}
$$

With these definitions

$$
\begin{aligned}
&\big[\Delta_v^D\,,\,\Delta_{v'}^D\big] = \Delta_{[v',v]}^D + \Delta_w^g + \Delta_{F_\mu}^g\ , \qquad [v',v]^i = v'^j\partial_j v^i - v^j\partial_j v'^i\ ,\\
&\big[\Delta_v^D\,,\,\Delta_{v'}'^D\big] - \big[\Delta_{v'}^D,\,\Delta_v'^D\big] = \Delta_{[v',v]}'^D + \Delta_{F'_\mu}^g\ ,\\
&w_i = 2H_{ijk}v'^j v^k\ , \qquad F_\mu = A_{\mu ij}v'^i v^j\ , \qquad F'_\mu = G_{ij}\big(v^i\partial'_\mu v'^j - v'^i\partial'_\mu v^j\big)\ .
\end{aligned}
\tag{4.10}
$$

The equations expressing invariance under the symmetries (4.3) and (4.6) are equivalent to relations between the finite local composite operators obtained by functional differentiation with respect to $\lambda(\phi,x)$, (4.8) corresponds to the equation of motion for $\nabla_\mu[\partial^\mu\phi^i u_i]$ while (4.9a,b) give operator identities for $\nabla_\mu i\epsilon^{\mu\nu}[\partial_\nu\phi^i w_i]$ and $\partial_\mu[F]$ respectively.

The discussion in section 2 was in terms of the vacuum energy functional W. Here, for reasons elucidated later, we apply similar arguments to S_0. The generator of local Weyl scaling Δ_σ^W is extended to

$$\Delta_\sigma^W = 2\int dv\,\sigma\Big(\gamma^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}} + T\cdot\frac{\delta}{\delta T}\Big) - \int dv\,\nabla^2\sigma\,\Phi\cdot\frac{\delta}{\delta T}\ , \tag{4.11}$$

so that $\Delta_\sigma^W S_0 = 0$, $[\Delta_\sigma^W, \Delta_{\sigma'}^W] = 0$. For β^G_{ij}, β^B_{ij} the appropriate β functions corresponding to the couplings G_{ij}, B_{ij}, which by power counting and invariance under (4.3) depend locally only on G, H and we assume $\bar\beta^G_{ij} = \beta^G_{ij}$, $\bar\beta^B_{ij} = -\beta^B_{ij}$, we define

$$\Delta_\sigma^\beta = \int dv\,\sigma\sum_{h=G,B}\beta^h\cdot\frac{\delta}{\delta h}\ . \tag{4.12}$$

To express the corresponding equation to (2.4) it is convenient to define

$$\mathcal{K}_\mu = \partial'_\mu g + i\epsilon_\mu{}^\nu g_{V_\nu} = \big(\partial'_\mu G_{ij},\, i\epsilon_\mu{}^\nu A_{\nu ij}\big)\ , \tag{4.13}$$

which forms a gauge invariant tangent vector to the space of dimensionless couplings. Then the general form, compatible with manifest invariance under (4.3) and the requirements of power counting, is

$$\begin{aligned}\Delta_\sigma^W S_0 = {}& \Delta_\sigma^\beta S_0\\ &+\int dv\,\sigma\Big(\big(O_i\mathcal{K}_\mu + i\epsilon_\mu{}^\nu\tilde O_i\mathcal{K}_\nu\big)\cdot\frac{\delta}{\delta V_{\mu i}} + \big(\tfrac12 R\,\beta^\Phi - UT' - X\big)\cdot\frac{\delta}{\delta T}\Big)S_0\\ &-\int dv\,\partial_\mu\sigma\Big(s_i\cdot\frac{\delta}{\delta V_{\mu i}} + i\epsilon^\mu{}_\nu\tilde s_i\cdot\frac{\delta}{\delta V_{\nu i}} + \big(W\mathcal{K}^\mu + i\epsilon^{\mu\nu}\tilde W\mathcal{K}_\nu\big)\cdot\frac{\delta}{\delta T}\Big)S_0\ ,\\ &\beta^\Phi = \Omega\Phi + \theta\ ,\qquad X = \tfrac12\big(\mathcal{K}^\mu\cdot\chi\cdot\mathcal{K}_\mu - i\epsilon^{\mu\nu}\mathcal{K}_\mu\cdot\tilde\chi\cdot\mathcal{K}_\nu\big)\ .\end{aligned} \tag{4.14}$$

Here O_i, $\tilde O_i$, W, $\tilde W$, U, Ω are appropriate linear operators while χ, $\tilde\chi$ are quadratic forms on vectors of the form $\mathcal{K}_\mu$, $\chi^T = \chi$, $\tilde\chi^T = -\tilde\chi$, each depending locally on G, H and restricted by $S_0(\phi,\lambda)^* = S_0(\phi,\bar\lambda)$. In β^Φ θ is a scalar and, along with the vectors s_i, $\tilde s_i$, is also formed from G, H with $\bar\theta = \theta$, $\bar s_i = s_i$ while $\bar{\tilde s}_i = -\tilde s_i$. Instead of the residual c number term as in (2.4) the extra contributions in (4.14) to the difference between $\gamma^{\mu\nu}T_{\mu\nu}$ and $\beta^i[\mathcal{O}_i]$, which in this case is $\frac12[\beta^G_{ij}\partial_\mu\phi^i\partial^\mu\phi^j] + \frac12[i\epsilon^{\mu\nu}\beta^B_{ij}\partial_\mu\phi^i\partial_\nu\phi^j]$, have a similar form but the coefficients may now depend on the dimensionless coordinate ϕ and so become scalar operators. Also additional terms containing the vector operators of dimension 1 present in the general σ model described by (4.1) are now allowed. By virtue of (4.9b) the expression for $(\Delta_\sigma^W - \Delta_\sigma^\beta)S_0$ is arbitrary up to contributions of the form $\Delta^g_{F_\mu}S_0$. Thus the result as written in (4.14) depends on $\nabla'^\mu V_{\mu i}$ through T'_i but this particular contribution can be removed by use of the relation (4.9b) at the cost of introducing other V_μ dependent terms. Moreover as a consequence of the second of eqs. (4.5) we can arrange that $\tilde{W}\mathcal{K}_\mu$ is independent of $A_{\nu ij}$ at the expense of changes in O_i, χ and U, in particular we require

$$\tilde{W} g_w = 0 \ , \tag{4.15}$$

and this leads to an arbitrariness under variations of the form

$$\delta\tilde{W}\partial'_\mu g = \partial'_\mu \omega \ , \quad \delta\tilde{O}_i \partial'_\mu g = -\partial_i \partial'_\mu \omega \ , \quad \bar{\omega} = -\omega \ . \tag{4.16}$$

For σ constant the extra contribution in (4.14) may also be identified with the β functions $\beta^V_{\mu i}$, β^T, thus, neglecting inhomogeneous terms containing V_μ or $\mathcal{K}_\mu$, $\beta^T = -U^i\partial_i T$.

Although the symmetry under (4.3) has been automatically satisfied the implications of invariance under diffeomorphisms on $\mathcal{M}$ are less explicit. To compute these requirements we use

$$\begin{aligned} \Delta^D_v \mathcal{K}_\mu &= (\mathcal{L}_v - v^k\partial_k)\mathcal{K}_\mu + g^D_{\partial'_\mu v} \ , \qquad & \Delta'^D_v \mathcal{K}_\mu &= i\epsilon_\mu{}^\nu g_{G\partial'_\nu v} \ , \\ \Delta^D_v T'_i &= (\mathcal{L}_v - v^k\partial_k)T'_i + A^\mu{}_{ij}\partial'_\mu v^j \ , \qquad & \Delta'^D_v T'_i &= -\nabla'^\mu (G_{ij}\partial'_\mu v^j) \ . \end{aligned} \tag{4.17}$$

Then if

$$\delta_v = (\mathcal{L}_v g)\cdot\frac{\partial}{\partial g} - \mathcal{L}_v \ , \tag{4.18}$$

we may obtain with the aid of (4.17)

$$\begin{aligned} &[\Delta^D_v, \Delta^\beta_\sigma] \sim \int dv\, \sigma \Big\{ \sum_{h=G,B} (\delta_v \beta^h)\cdot\frac{\delta}{\delta h} - i\epsilon_\mu{}^\nu \big(\beta^B_{ij}\partial'_\nu v^j\big)\cdot\frac{\delta}{\delta V_{\mu i}} \Big\} \ , \\ &[\Delta'^D_v, \Delta^\beta_\sigma] = -\int dv\, \sigma\, \big(\beta^G_{ij}\partial'_\nu v^j\big)\cdot\frac{\delta}{\delta V_{\mu i}} \ , \\ &[\Delta^D_v, \int dv\, \sigma\, (O_i\mathcal{K}_\mu)\cdot\frac{\delta}{\delta V_{\mu i}}] = \int dv\, \sigma \Big\{ \big((\delta_v O_i)\mathcal{K}_\mu\big)\cdot\frac{\delta}{\delta V_{\mu i}} + \big(O_i g^D_{\partial'_\mu v}\big)\cdot\frac{\delta}{\delta V_{\mu i}} \\ &[\Delta'^D_v, \int dv\, \sigma\, (O_i\mathcal{K}_\mu)\cdot\frac{\delta}{\delta V_{\mu i}}] \sim \int dv\, \sigma \Big\{ i\epsilon_\mu{}^\nu \big(O_i g_{G\partial'_\nu v}\big)\cdot\frac{\delta}{\delta V_{\mu i}} - \big(\partial'^\mu v^i O_i \mathcal{K}_\mu\big)\cdot\frac{\delta}{\delta T} \Big\} \ , \\ &[\Delta^D_v, \int dv\, \partial_\mu\sigma\, (W\mathcal{K}^\mu)\cdot\frac{\delta}{\delta T}] = \int dv\, \partial_\mu\sigma \Big\{ \big((\delta_v W)\mathcal{K}^\mu\big)\cdot\frac{\delta}{\delta T} + \big(W g^D_{\partial'^\mu v}\big)\cdot\frac{\delta}{\delta T} \Big\} \ , \\ &[\Delta^D_v, \int dv\, \sigma\, (UT')\cdot\frac{\delta}{\delta T}] = \int dv\, \sigma \Big\{ \big((\delta_v U)T'\big)\cdot\frac{\delta}{\delta T} + \big(U(A^\mu\partial'_\mu v)\big)\cdot\frac{\delta}{\delta T} \Big\} \ , \\ &[\Delta'^D_v, \int dv\, \sigma\, (UT')\cdot\frac{\delta}{\delta T}] \sim \int dv\, \sigma \Big\{ \big(\partial_i(UG\partial'_\mu v)\big)\cdot\frac{\delta}{\delta V_{\mu i}} + \big(\partial'^\mu U(G\partial'_\mu v)\big)\cdot\frac{\delta}{\delta T} \Big\} \\ &\qquad\qquad + \int dv\, \partial_\mu\sigma\, \big(U(G\partial'^\mu v)\big)\cdot\frac{\delta}{\delta T} \ , \\ &[\Delta^D_v, \int dv\, \partial_\mu\sigma\, s_i\cdot\frac{\delta}{\delta V_{\mu i}}] \sim \int dv\, \partial_\mu\sigma \Big\{ \big(\delta_v s_i\big)\cdot\frac{\delta}{\delta V_{\mu i}} - \big(\partial'^\mu v^i s_i\big)\cdot\frac{\delta}{\delta T} \Big\} \ , \end{aligned} \tag{4.19}$$

where $\sim$ denotes equality up terms vanishing when acting on S_0 as a consequence of (4.9a,b). Hence applying $\Delta_v^D + \Delta_v'^D$ to both sides of (4.14) and using the results in (4.19) and (4.8) implies various identities. The requirement that the terms involving δ_v vanish is just the condition that the various β functions and other operators appearing in (4.14) are tensors constructed from G_{ij}, B_{ij} and the remaining terms depending on $\partial'_\mu v$ give the relations

$$\begin{aligned}
\beta^G_{ij}\partial'_\mu v^j + i\epsilon_\mu{}^\nu \beta^B_{ij}\partial'_\nu v^j &= O_i \mathcal{J}_\mu + i\epsilon_\mu{}^\nu \tilde{O}_i \mathcal{J}_\nu - \partial_i\big(U(G\partial'_\mu v)\big) + i\epsilon_\mu{}^\nu \partial_i (Q\partial'_\nu v) \ , &(4.20a)\\
\partial'^\mu v^i \big(O_i \mathcal{K}_\mu + i\epsilon_\mu{}^\nu \tilde{O}_i \mathcal{K}_\nu\big)& \\
= -\mathcal{K}^\mu{\cdot}\chi{\cdot}\mathcal{J}_\mu + i\epsilon^{\mu\nu}\mathcal{K}_\mu{\cdot}\tilde{\chi}{\cdot}\mathcal{J}_\nu &- \partial'^\mu U(G\partial'_\mu v) - U(A_\mu \partial'^\mu v) + i\epsilon^{\mu\nu}\partial'_\mu Q \partial'_\nu v \ , &(4.20b)\\
\partial'_\mu v^i s_i + i\epsilon_\mu{}^\nu \partial'_\nu v^i \tilde{s}_i &= W\mathcal{J}_\mu + i\epsilon_\mu{}^\nu \tilde{W}\mathcal{J}_\nu + U(G\partial'_\mu v) - i\epsilon_\mu{}^\nu Q\partial'_\nu v \ , &(4.20c)\\
\mathcal{J}_\mu &= g^D_{\partial'_\mu v} + i\epsilon_\mu{}^\nu g_{G\partial'_\nu v} \ ,
\end{aligned}$$

where the terms depending on the linear operator $Q = -\bar{Q}$ arise as a consequence of the possible arbitrariness expressed by (4.9a). From (4.20a,c)

$$\begin{aligned}
\beta^G_{ij} v^j &= O_i g^D_v + \tilde{O}_i g_{Gv} - \partial_i\big(U(Gv)\big) \ , &(4.21a)\\
\beta^B_{ij} v^j &= O_i g_{Gv} + \tilde{O}_i g^D_v + \partial_i\big(Qv\big) \ , \quad Qv = W g_{Gv} + \tilde{W} g^D_v - v^j \tilde{s}_j \ , &(4.21b)\\
v^i s_i - U(Gv) &= W g^D_v \ , &(4.21c)
\end{aligned}$$

using (4.15), while (4.20b) gives, with Q determined by (4.21b),

$$\begin{aligned}
v^i O_i \delta g &= -\, g^D_v{\cdot}\chi{\cdot}\delta g - g_{Gv}{\cdot}\tilde{\chi}{\cdot}\delta g - \delta U(Gv) \ , \\
v^i \tilde{O}_i \delta g &= g_{Gv}{\cdot}\chi{\cdot}\delta g + g^D_v{\cdot}\tilde{\chi}{\cdot}\delta g - U(\delta B v) - \delta Q v \ .
\end{aligned} \qquad (4.22)$$

Note that (4.22) with (4.21a,b) implies $v^{[i}v'^{j]}\beta^G_{ij} = v^{(i}v'^{j)}\beta^B_{ij} = 0$ as expected.

The derivation of the Weyl consistency conditions which arise from the requirement

$$\big[\Delta^W_\sigma - \Delta^\beta_\sigma, \, \Delta^W_{\sigma'} - \Delta^\beta_{\sigma'}\big] S_0 = 0 \ , \qquad (4.23)$$

follows in a similar fashion to section 2. Using

$$\Delta^\beta_\sigma \mathcal{K}_\mu - \int dv \, \partial_\nu \sigma \Big(s_i {\cdot} \frac{\delta}{\delta V_{\nu i}} + i\epsilon^\nu{}_\rho \tilde{s}_i {\cdot} \frac{\delta}{\delta V_{\rho i}} \Big) \mathcal{K}_\mu = \sigma \, \mathcal{D}_\beta \mathcal{K}_\mu + \partial_\mu \sigma \, (\beta^g + g_{\tilde{s}}) - i\epsilon_\mu{}^\nu \partial_\nu \sigma \, g_s \ , \quad (4.24)$$

where

$$\beta^g = \big(\beta^G_{ij}, \, \beta^B_{ij}\big) \ , \qquad \mathcal{D}_\beta = \sum_{h=G,B} \beta^h {\cdot} \frac{\partial}{\partial h} \ ,$$

then we may derive for various different contributions arising from (4.14)

$$\big[\Delta^W_\sigma , \int dv \, \sigma' \big(\tfrac{1}{2}\beta^\Phi R - UT'\big){\cdot}\frac{\delta}{\delta T}\big] - (\sigma \leftrightarrow \sigma')$$

$$
\begin{aligned}
&\sim \int dv\, \xi_\mu \Big\{ \big(\partial_i \theta\big)\cdot \frac{\delta}{\delta V_{\mu i}} + \partial'^\mu \theta \cdot \frac{\delta}{\delta T}\Big\} - \int dv\, \nabla^\mu \xi_\mu \big((\Omega + U\partial)\Phi\big)\cdot \frac{\delta}{\delta T} \ , \\
&\Big[\int dv\, \sigma \big(UT'\big)\cdot \frac{\delta}{\delta T}, \int dv\, \partial_\mu \sigma' \big(W\mathcal{K}^\mu\big)\cdot \frac{\delta}{\delta T}\Big] - (\sigma \leftrightarrow \sigma') = - \int dv\, \xi_\mu \,\big(U\partial (W\mathcal{K}^\mu)\big)\cdot \frac{\delta}{\delta T} \ , \\
&\Big[\Delta^\beta_\sigma - \int dv\, \partial_\nu \sigma \Big(s_i \cdot \frac{\delta}{\delta V_{\nu i}} + i\epsilon^\nu{}_\rho \tilde{s}_i \cdot \frac{\delta}{\delta V_{\rho i}}\Big), \int dv\, \partial_\mu \sigma' \big(W\mathcal{K}^\mu\big)\cdot \frac{\delta}{\delta T}\Big] - (\sigma \leftrightarrow \sigma') \\
&\qquad\qquad = \int dv\, \xi_\mu \,\big(\mathcal{D}_\beta (W\mathcal{K}^\mu)\big)\cdot \frac{\delta}{\delta T} - \int dv\, 2i\epsilon^{\mu\nu} \partial_\mu \sigma' \partial_\nu \sigma \,\big(W g_s\big)\cdot \frac{\delta}{\delta T} \ , \qquad (4.25) \\
&\Big[\int dv\, \sigma \,\big(O_i \mathcal{K}_\mu\big)\cdot \frac{\delta}{\delta V_{\mu i}}, \int dv\, \sigma' \big(UT'\big)\cdot \frac{\delta}{\delta T}\Big] - (\sigma \leftrightarrow \sigma') = \int dv\, \xi_\mu \,\big(UO\mathcal{K}^\mu\big)\cdot \frac{\delta}{\delta T} \ , \\
&\Big[\int dv\, \partial_\mu \sigma \, i\epsilon^\mu{}_\nu \tilde{s}_i \cdot \frac{\delta}{\delta V_{\nu i}}, \int dv\, \sigma' \big(UT'\big)\cdot \frac{\delta}{\delta T}\Big] - (\sigma \leftrightarrow \sigma') \\
&\quad \sim - \int dv\, \xi_\mu \, i\epsilon^\mu{}_\nu \Big\{ \big(\partial_i (U\tilde{s})\big)\cdot \frac{\delta}{\delta V_{\nu i}} + \big(\partial'^\nu U\tilde{s}\big)\cdot \frac{\delta}{\delta T}\Big\} - \int dv\, 2i\epsilon^{\mu\nu} \partial_\mu \sigma' \partial_\nu \sigma \,\big(U\tilde{s}\big)\cdot \frac{\delta}{\delta T} \ , \\
&\qquad\qquad \xi_\mu = \sigma \partial_\mu \sigma' - \sigma' \partial_\mu \sigma \ .
\end{aligned}
$$

From the Φ dependent terms we find

$$
\Omega = - U\partial \ , \qquad (4.26)
$$

and using

$$
\int dv\, 2i\epsilon^{\mu\nu} \partial_\mu \sigma' \partial_\nu \sigma \, \tilde{\rho}\cdot \frac{\delta}{\delta T} \sim \int dv\, i\epsilon_\mu{}^\nu \xi_\nu \Big\{ \big(\partial_i \tilde{\rho}\big)\cdot \frac{\delta}{\delta V_{\mu i}} + \big(\partial'^\mu \tilde{\rho}\big)\cdot \frac{\delta}{\delta T}\Big\} \ , \qquad (4.27)
$$

we obtain from the terms involving $\delta/\delta V_\mu$

$$
\begin{aligned}
\partial_i \big(\theta + Us\big) &= \, - O_i (\beta^g + g_{\tilde{s}}) + \tilde{O}_i g_s - \mathcal{D}_\beta s_i \ , \qquad (4.28a) \\
\partial_i \big(-W g_s + \tilde{W}\beta^g\big) &= O_i g_s - \tilde{O}_i (\beta^g + g_{\tilde{s}}) + \mathcal{D}_\beta \tilde{s}_i \ , \qquad (4.28b)
\end{aligned}
$$

and also for the remainder proportional to $\delta/\delta T$

$$
\begin{aligned}
&\partial'_\mu \big(\theta + Us\big) - i\epsilon_\mu{}^\nu \partial'_\nu \big(-W g_s + \tilde{W}\beta^g\big) \\
&\quad = (\beta^g + g_{\tilde{s}})\cdot \chi \cdot \mathcal{K}_\mu - g_s \cdot \tilde{\chi}\cdot \mathcal{K}_\mu + i\epsilon_\mu{}^\nu \big(g_s \cdot \chi \cdot \mathcal{K}_\nu - (\beta^g + g_{\tilde{s}})\cdot \tilde{\chi}\cdot \mathcal{K}_\nu\big) \\
&\qquad - \big(\mathcal{D}_\beta + U\partial\big)\big(W\mathcal{K}_\mu + i\epsilon_\mu{}^\nu \tilde{W}\mathcal{K}_\nu\big) - W g_{\tilde{O}\mathcal{K}_\mu} - i\epsilon_\mu{}^\nu \tilde{W} g_{O\mathcal{K}_\nu} \qquad (4.29) \\
&\qquad - U\big(O\mathcal{K}_\mu + i\epsilon_\mu{}^\nu \tilde{O}\mathcal{K}_\nu - \partial'_\mu s - i\epsilon_\mu{}^\nu \partial'_\nu \tilde{s}\big) \ .
\end{aligned}
$$

It is important to recognise that (4.14) provides a finite local operator expression for the trace of the energy momentum tensor $\gamma^{\mu\nu}T_{\mu\nu}$ which is independent of ambiguities in the definition of the β functions. Using (4.8) and (4.9a,b) the terms involving $\partial_\mu \sigma$ may be eliminated at the expense of modifying $\beta^\lambda \to B^\lambda$. For simplicity when V_μ and $\partial'_\mu g$ are both set to zero after the action of the functional derivatives on S_0 (4.14) gives

$$\begin{aligned}\gamma^{\mu\nu}T_{\mu\nu} = {}& \sum_{h=G,B} \beta^h\cdot\frac{\delta}{\delta h}S_0 + \tfrac12 R\,\beta^\Phi\cdot\frac{\delta}{\delta T}S_0 + (\beta^T - 2T)\cdot\frac{\delta}{\delta T}S_0 \\ & + \nabla_\mu\Big((s_i+\partial_i\Phi)\cdot\frac{\delta}{\delta V_{\mu i}} + i\epsilon^\mu{}_\nu\tilde s_i\cdot\frac{\delta}{\delta V_{\nu i}}\Big)S_0\Big|_{V_\mu=0} \\ = {}& \sum_{h=G,B} B^h\cdot\frac{\delta}{\delta h}S_0 + \tfrac12 R\,B^\Phi\cdot\frac{\delta}{\delta T}S_0 + (B^T-2T)\cdot\frac{\delta}{\delta T}S_0 - d^i\frac{\delta}{\delta\phi^i}S_0\ ,\end{aligned}\tag{4.30}$$

where, using the equation of motion (4.8) with (4.6) and (4.9a,b),

$$\begin{aligned} &B^\lambda = \big(\beta^\lambda + \lambda_d^D + \lambda_{\tilde s,0}\big)\big|_{V_\mu=\partial'_\mu g=0}\ , \qquad d^i = G^{ij}(s_j+\partial_j\Phi)\ , \\ &B^G_{ij} = \beta^G_{ij} + \mathcal{L}_d G_{ij}\ , \qquad\qquad B^B_{ij} = \beta^B_{ij} + 2H_{ijk}d^k + (\mathrm{d}\tilde s)_{ij}\ , \\ &B^\Phi = \beta^\Phi + \mathcal{L}_d\Phi = \Delta\Phi + \theta\ , \qquad B^T = \beta^T + \mathcal{L}_d T = \Delta T\ , \quad \Delta = (d-U)\partial\ .\end{aligned}\tag{4.31}$$

These particular additional terms in going from β^λ to B^λ are a consequence of the freedom in determining the β functions resulting from invariance under diffeomorphisms and the gauge symmetries (4.3), thus β^B_{ij} is arbitrary up to $\delta\beta^B_{ij} = -(\mathrm{d}w)_{ij}$ but in (4.14) this requires $\delta\tilde s_i = w_i$, $\delta\tilde O_i\mathcal{K}_\mu = \partial'_\mu w_i$ and hence this cancels in B^B_{ij}. The essential conditions for local scale invariance are then

$$B^G_{ij} = B^B_{ij} = 0\ , \qquad \Delta T - 2T = 0\ .\tag{4.32}$$

From the definition of B^Φ in (4.31) with (4.28a) and (4.21a) we obtain

$$\begin{aligned}\partial_i B^\Phi &= \partial_i\big(\theta + Us + d^j\partial_j\Phi - UGd\big) \\ &= -O_iB^g + \mathcal{L}_d\partial_i\Phi - \mathcal{D}_\beta s_i + \beta^G_{ij}d^j \\ &= -O_iB^g - \mathcal{D}_B s_i + B^G_{ij}d^j\ , \qquad B^g = \big(B^G_{ij},\, B^B_{ij}\big)\ .\end{aligned}\tag{4.33}$$

This is just the Curci-Paffuti relation [21] showing that B^Φ is ϕ independent at a fixed point when B^G_{ij}, B^B_{ij} are zero.* From (4.29) and (4.22) we may also obtain for arbitrary variations δg

$$\begin{aligned}\delta\beta^\Phi &= \delta\big(\theta + Us - UGd\big) \\ &= B^g\cdot\chi\cdot\delta g + \big(d^iO_i - UO\big)\delta g - (\mathcal{D}_\beta + U\partial)(W\delta g) - Wg_{\tilde O\delta g}\ .\end{aligned}\tag{4.34}$$

Using now

$$\begin{aligned}\mathcal{D}_\beta(W\delta g) &= \mathcal{D}_B(W\delta g) - d^i\partial_i(W\delta g) - W\big(g^D_{\delta d} + g_{\delta Bd} + g_{\delta\tilde s}\big)\ , \\ Wg^D_{\delta d} &= -U(G\delta d) + d^iG_{ij}\delta d^j - \delta d^i\partial_i\Phi\ ,\end{aligned}$$

* An essentially similar derivation of this relation is ascribed to Polchinski by de Alwis [25].

from (4.15) and (4.21c), then (4.343) becomes

$$
\begin{gathered}
\delta B^{\Phi} = B^{g}\cdot\chi\cdot\delta g - \mathcal{D}_B(W\delta g) + (d-U)(\rho\delta g) - W g_{\tilde\rho\delta g} \ , \\
\rho_i\delta g = O_i\delta g + G_{ij}\delta d^j + \partial_i(W\delta g) \ , \qquad \tilde\rho_i\delta g = \tilde O_i\delta g - \delta B_{ij}d^j - \delta\tilde s_i + \partial_i(\tilde W\delta g) \ ,
\end{gathered}
\tag{4.35}
$$

where ρ, $\tilde\rho$ are linear operators on tangent vectors to the space of couplings.† With these definitions (4.33) and also the corresponding equation from (4.28b) become

$$
\partial_i B^{\Phi} = -\rho_i B^g + \partial_i(WB^g) \ , \qquad \tilde\rho_i B^g = 0 \ , \tag{4.36}
$$

while (4.21a,b) may be written more simply as

$$
\begin{aligned}
B^G_{ij}v^j &= \rho_i g^D_v + \tilde\rho_i g_{Gv} + 2\,\partial_i\mathcal{L}_v\Phi \ , \\
B^B_{ij}v^j &= \rho_i g_{Gv} + \tilde\rho_i g^D_v \ .
\end{aligned}
\tag{4.37}
$$

The resulting equation (4.35) is the analog of (2.7) for general non linear σ models and has been used by us earlier in order to construct an effective action, to all orders, which is stationary when B^G, B^B, B^Φ are zero. Defining, as in (2.8),

$$
\mathcal{C} = B^{\Phi} + WB^g \ , \tag{4.38}
$$

then under variations of Φ, using (4.21c),

$$
\delta\mathcal{C} = 2\Delta\delta\Phi \ , \tag{4.39}
$$

with Δ defined in (4.31), and hence

$$
I = \int d^D\phi\, J\,\mathcal{C} \ , \tag{4.40}
$$

satisfies, from (4.35) and (4.39),

$$
\delta I = \mathrm{O}\big(B^G,\, B^B,\, B^\Phi\big) \ , \tag{4.41}
$$

so long as the measure J is constructed to satisfy, for any $u_i(\phi)$,

$$
\int d^D\phi\, J\,(d-U)u = \mathrm{O}\big(B^G,\, B^B\big) \ . \tag{4.42}
$$

and W is constrained to satisfy $Wg_w = 0$ for arbitrary w_i.

From (4.35), (4.39) and (4.33) we may also obtain

$$
\sum_{h=G,B,\Phi} B^h\cdot\frac{\partial}{\partial h}\mathcal{C} - \Delta\mathcal{C} = B^g\cdot\chi\cdot B^g \ , \tag{4.43}
$$

† Note that $B^V_{\mu i} = \rho_i\mathcal{K}_\mu + i\epsilon_\mu{}^\nu\tilde\rho_i\mathcal{K}_\nu + 2\,\partial_i\partial'_\mu\Phi$.

which is the corresponding version of (2.9). Δ appears here as the differential operator defining the anomalous dimension for the scalar $\mathcal{C}(\phi)$. At one loop Δ is essentially the laplacian on $\mathcal{M}$ and for $\mathcal{M}$ compact Δ may be expected to have a discrete spectrum with in general constant functions on $\mathcal{M}$ corresponding to the eigenvalue zero. For σ models on compact target spaces $\mathcal{M}$ it should be possible to project (4.43) on to such constant modes and then (4.43) should be equivalent to the Zamolodchikov equation defining a monotonic renormalisation flow of the c-function. On a non compact manifold $\mathcal{M}$, as appropriate in string theory if the vanishing of the β function is to be interpretated as an extension of the Einstein equations for gravity, Δ has a continuous spectrum so that derivation of the conventional c-theorem, with its consequences of a monotonic flow of the couplings, becomes problematic [26]. If the results such as (4.43) are rephrased in terms of expectation values, as was done in section 2, then there are potential difficulties in general since $\langle \mathcal{C}(\phi)\rangle$ has infra-red divergences on flat space necessitating the introduction of a further scale as an infra-red cutoff [27,28].

At one loop the results of previous calculations, which should be independent of the regularisation scheme to this order, are given by (for $\ell = 4\pi$)

$$\begin{aligned}
\beta^G_{ij} &= 2\big(R_{ij} - H_i{}^{k\ell}H_{jk\ell}\big) \, , \qquad \beta^B_{ij} = -2\nabla^k H_{ijk} \, , \qquad \beta^\Phi = -\nabla^2\Phi + \tfrac{1}{3}D \, , \\
O_i\delta g &= \nabla^j \delta G_{ij} - \tfrac{1}{2}\,\partial_i\big(G^{jk}\delta G_{jk}\big) - H_i{}^{jk}\delta B_{jk} \, , \qquad \tilde O_i \delta g = \nabla^j \delta B_{ij} \, , \\
Uw &= \nabla^i w_i \, , \qquad W\delta g = -\tfrac{1}{2}G^{ij}\delta G_{ij} \, , \qquad \tilde W = 0 \, , \\
\delta g{\cdot}\chi{\cdot}\delta g &= \tfrac{1}{2}G^{ik}G^{j\ell}\big(\delta G_{ij}\,\delta G_{k\ell} + \delta B_{ij}\,\delta B_{k\ell}\big) \, , \qquad s_i = \tilde s_i = 0 \, ,
\end{aligned} \tag{4.44}$$

with ∇_i the covariant derivative defined by the Christoffel connection formed from the metric G_{ij} and R_{ij} the associated Ricci tensor. With other quantities zero to this order these results obey the relations derived in general in this section and from (4.38), (4.42)

$$\mathcal{C} = \tfrac{1}{3}D + \partial^i\Phi\partial_i\Phi - 2\nabla^2\Phi - R + \tfrac{1}{3}H^{ijk}H_{ijk} \, , \quad J = \sqrt{G}\, e^{-\Phi} \, , \tag{4.45}$$

with here R the scalar curvature on $\mathcal{M}$. Clearly from (4.44) $\delta g{\cdot}\chi{\cdot}\delta g$ is positive at lowest order. The action S in (4.1), with fields defined on a Riemmanian two dimensional space with positive definite metric, is not real but satisfies the requirement $S^\theta = S$ where the operation θ includes as well as complex conjugation a coordinate change inducing a reversal of orientation, such as $(x^1,x^2)^\theta = (x^1,-x^2)$. This reality condition is necessary for ensuring unitarity of the associated quantum field theory and translates, at least formally, into reflection positivity for correlation functions $\langle \mathcal{O}(x)\mathcal{O}(x')^\theta\rangle$ which is sufficient to ensure that a positive definite metric on the space of couplings g may be defined to all orders in the general bosonic σ model.

Using dimensional regularisation and minimal subtraction calculations have been extended to two loops [13] in general and to four [15] when $B_{ij}=0$. The action proposed by Tseytlin [22] is obtained if there is a choice of reparameterisation of G_{ij}, B_{ij} and of the additional arbitrariness in the renormalisation expressions appearing in (4.14), such as can be obtained by a local redefinition of $V_{\mu i}$, T, which ensures that the one loop result for W in (4.44) and $\tilde W = 0$, and hence also for $d-U$ and J in (4.45), are valid to all orders in the perturbation expansion.

5. Non Linear σ Models, Boundary Terms

From the point of view of string theory it is natural to consider non linear σ models on general two dimensional spaces with a boundary, corresponding to open strings. Besides the usual action (4.1) the inclusion of the open string modes in this approach is realised by an additional contribution $\hat{S}$ involving fields restricted to the boundary of the two dimensional space on which the closed string fields are defined. For the renormalisable σ model of the previous section it is natural to restrict $\hat{S}$ to depend only on the vector A_i, representing the massless spin 1 particle of the open string, and the scalars $\hat{T}$, $\hat{\Phi}$ for the open string spinless tachyon, dilaton respectively,

$$\hat{S} = \int ds \left(i\dot{x}^\mu A_i \partial_\mu \phi^i + \hat{\Phi}\, K + \hat{T} \right) , \tag{5.1}$$

with s the arc length along the boundary, K the extrinsic curvature and

$$\dot{x}^\mu = \frac{dx^\mu}{ds} , \qquad n^\mu \epsilon_\mu{}^\nu = \dot{x}^\nu , \tag{5.2}$$

for n^μ the unit inward normal to the boundary. The additional couplings defined on the boundary are denoted by

$$\hat{\lambda} = (A_i,\, \hat{\Phi},\, \hat{T}) , \tag{5.3}$$

are also assumed to depend arbitrarily on $x(s)$ on the boundary as well as being a vector, scalars on the target manifold of the σ model $\hat{\lambda}(\phi, x)$. The bar operation is now extended by $\bar{A}_i = -A_i$, with $\hat{\Phi}$, $\hat{T}$ unchanged, so that $\hat{S}(\phi, \hat{\lambda})^* = \hat{S}(\phi, \bar{\hat{\lambda}})$. In $\hat{S}$ there is no dependence on $\partial_n \phi^i = n^\mu \partial_\mu \phi^i$, as is necessary for application to strings, and we further assume no such terms are required when using the appropriate boundary conditions for the quantum fields as counterterms in a loop expansion, a justification is given in the appendix.

The symmetry transformations (4.3) are now extended by

$$\delta\hat{\lambda} = \hat{\lambda}_{w,F_\mu} = \left(w_i,\, 0,\, n^\mu F_\mu \right) , \tag{5.4}$$

to ensure invariance of $S + \hat{S}$. In addition there is a further gauge symmetry

$$\delta\hat{\lambda} = \hat{\lambda}_\Lambda = \left(\partial_i \Lambda,\, 0,\, i\dot{x}^\mu \partial'_\mu \Lambda \right) , \qquad \bar{\Lambda} = -\Lambda . \tag{5.5}$$

As in (4.4) we may define invariants under these gauge symmetries, restricted to the boundary, by

$$F_{ij} = (\mathrm{d}A)_{ij} - B_{ij} , \qquad \hat{T}'_i = \partial_i \hat{T} - n^\mu V_{\mu i} - i\dot{x}^\mu \partial'_\mu A_i , \tag{5.6}$$

where

$$(\mathrm{d}\hat{T}')_{ij} = -n^\mu A_{\mu ij} - i\dot{x}^\mu \partial'_\mu F_{ij} , \qquad (\mathrm{d}F)_{ijk} = -2H_{ijk} . \tag{5.7}$$

For diffeomorphisms, or reparameterisations of the coordinates on $\mathcal{M}$, then in addition to (4.6) we take

$$\delta\hat{\lambda} = \hat{\lambda}_v^D = \big(-F_{ij}v^j ,\, \mathcal{L}_v\hat{\Phi},\, \hat{T}'_i v^i\big) \ , \tag{5.8}$$

since, with notation as in (4.7),

$$\begin{aligned} \mathcal{L}_v A_i = {} & -F_{ij}v^j + \partial_i(A_j v^j) - \tilde{u}_i \ , \\ \mathcal{L}_v\hat{T} = {} & \hat{T}'_i v^i + i\dot{x}^\mu\partial'_\mu(A_i v^i) + n^\mu V_{\mu i}v^i - i\dot{x}^\mu A_i\partial'_\mu v^i \ . \end{aligned} \tag{5.9}$$

The generators of diffeomorphisms and gauge symmetries are now extended beyond the expressions given in (4.8) and (4.9a,b) to include boundary terms,

$$\begin{aligned} \Delta_v^D = {} & \ldots + \int ds\, \hat{\lambda}_v^D\cdot\frac{\delta}{\delta\hat{\lambda}} \ , \\ \Delta_w^g = {} & \ldots + \int ds\, w_i\cdot\frac{\delta}{\delta A_i} \ , \qquad \Delta_{F_\mu}^g = \ldots + \int ds\, n^\mu F_\mu\cdot\frac{\delta}{\delta\hat{T}} \ , \end{aligned} \tag{5.10}$$

with in addition

$$\hat{\Delta}_\Lambda^g = \int ds\,\Big(\partial_i\Lambda\cdot\frac{\delta}{\delta A_i} + i\dot{x}^\mu\partial'_\mu\Lambda\cdot\frac{\delta}{\delta\hat{T}}\Big) \ . \tag{5.11}$$

With these expressions (4.10) is modified to

$$\big[\Delta_v^D ,\, \Delta_{v'}^D\big] = \Delta_{[v',v]}^D + \Delta_w^g + \Delta_{F_\mu}^g + \hat{\Delta}_\Lambda^g \ , \qquad \Lambda = F_{ij}v'^i v^j \ . \tag{5.12}$$

As before the regularisation procedure is assumed to preserve these symmetries so that the bare action S_0, which now includes boundary contributions of the same form as (5.1), is required to satisfy

$$(\Delta_v^D + \Delta_v'^D)S_0 = \Delta_w^g S_0 = \Delta_{F_\mu}^g S_0 = \hat{\Delta}_\Lambda^g S_0 = 0 \ . \tag{5.13}$$

Furthermore we also require $S_0(\phi,\lambda,\hat{\lambda})^* = S_0(\phi,\bar{\lambda},\bar{\hat{\lambda}})$.

The essential local renormalisation group equation is now extended from (4.14) by including boundary terms similar (2.21) so that it becomes

$$\begin{aligned} \Delta_\sigma^W S_0 = {} & \Big(\Delta_\sigma^\beta + \int ds\,\beta_i^A\cdot\frac{\delta}{\delta A_i}\Big)S_0 + \ldots \\ & + \int ds\,\sigma\big(K\,\beta^{\hat{\Phi}} - \hat{U}\hat{T}' + n^\mu\omega\mathcal{K}_\mu + i\dot{x}^\mu\tilde{\omega}\mathcal{K}_\mu\big)\cdot\frac{\delta}{\delta\hat{T}}S_0 \\ & - \int ds\,\partial_\mu\sigma\big(n^\mu\epsilon + i\dot{x}^\mu\tilde{\epsilon}\big)\cdot\frac{\delta}{\delta\hat{T}}S_0 \ , \qquad \beta^{\hat{\Phi}} = \hat{\Omega}\hat{\Phi} + \hat{\theta} \ , \\ \Delta_\sigma^W = {} & \ldots + \int ds\,\sigma\,\hat{T}\cdot\frac{\delta}{\delta\hat{T}} - \int ds\,\partial_\mu\sigma\, n^\mu\hat{\Phi}\cdot\frac{\delta}{\delta\hat{T}} \ , \end{aligned} \tag{5.14}$$

with $\hat{U}$, ω, $\tilde{\omega}$, $\hat{\Omega}$ appropriate linear operators and like the β function β_i^A and the scalars ϵ, $\tilde{\epsilon}$, $\hat{\theta}$ constructed from G, H and also F, with ϵ, $\hat{\theta}$ invariant under the bar operation while $\bar{\tilde{\epsilon}} = -\tilde{\epsilon}$. The β function β_i^A has been calculated [29,30,31] but the other boundary terms in (5.14) have not previously been determined. Neglecting V_μ and the x dependence of the couplings as usual we may identify $\beta^{\hat{T}} = -\hat{U}\partial\hat{T}$. Just as in (4.31) we may use (5.13) to eliminate the $\tilde{\epsilon}$ term in (5.14), as well as s_i, $\tilde{s}_i$ at the expense of modifying the β functions

$$\begin{aligned} &\beta_i^A \to B_i^A = \beta_i^A + \tilde{s}_i - F_{ij}d^j + \partial_i\tilde{\epsilon} \ , \\ &\beta^{\hat{T}} \to B^{\hat{T}} = (d - \hat{U})\partial\hat{T} \ , \qquad \beta^{\hat{\Phi}} \to B^{\hat{\Phi}} = \beta^{\hat{\Phi}} + d^i\partial_i\hat{\Phi} \ . \end{aligned} \tag{5.15}$$

As before B_i^A, $B^{\hat{T}}$ are independent of the ambiguities in the definitions of the β functions arising from the gauge invariances of the basic theory.

By an extension of the arguments of the previous section there are also further constraints on the boundary terms in (5.14). When the generator of diffeomorphisms $\Delta_v^D + \Delta_v'^D$ is applied to both sides of (5.14) and invariance of S_0 as in (5.13) is imposed then the expressions for the resulting commutators, such as are given in (4.19), may now have additional terms restricted to integrals over the boundary. For calculation the important contributions are given by

$$\begin{aligned} &[\Delta_v^D , \Delta_\sigma^\beta] \sim 0 \ , \\ &[\Delta_v^D + \Delta_v'^D , \int dv\, \sigma\, (O_i\mathcal{K}_\mu)\cdot\frac{\delta}{\delta V_{\mu i}}] \sim 0 \ , \\ &[\Delta_v^D , \int dv\, \partial_\mu\sigma\, s_i \cdot \frac{\delta}{\delta V_{\mu i}}] \sim 0 \ , \\ &[\Delta_v'^D , \int dv\, \sigma\, (UT')\cdot\frac{\delta}{\delta T}] \sim \int ds\, \sigma\, n^\mu \big(U(G\partial'_\mu v)\big)\cdot\frac{\delta}{\delta \hat{T}} \ , \\ &[\Delta_v'^D , \int dv\, \sigma\, (\hat{U}\hat{T}')\cdot\frac{\delta}{\delta \hat{T}}] = -\int ds\, \sigma\, n^\mu \big(\hat{U}(G\partial'_\mu v)\big)\cdot\frac{\delta}{\delta \hat{T}} \ , \\ &[\Delta_v^D , \int dv\, \sigma\, (\hat{U}\hat{T}')\cdot\frac{\delta}{\delta \hat{T}}] = \int ds\, \sigma\big((\delta_v\hat{U})\hat{T}' + i\dot{x}^\mu\hat{U}(F\partial'_\mu v)\big)\cdot\frac{\delta}{\delta \hat{T}} \ , \\ &[\Delta_v^D , \int dv\, \sigma\, \beta_i^A \cdot\frac{\delta}{\delta A_i}] \sim \int ds\, \sigma\Big(\big(\delta_v\beta_i^A\big)\cdot\frac{\delta}{\delta A_i} - i\dot{x}^\mu\big(\beta_i^A\partial'_\mu v^i\big)\cdot\frac{\delta}{\delta \hat{T}}\Big) \ , \end{aligned} \tag{5.16}$$

where δ_v is the generalisation of (4.18) to also include diffeomorphisms acting on A. Apart from conditions such as $\delta_v\beta_i^A = 0$, which just requires that β_i^A is an appropriate tensor constructed from G, B, A, collecting the terms involving $\partial'_\mu v$ gives

$$n^\mu(U - \hat{U})G\partial'_\mu v + i\dot{x}^\mu\beta_i^A\partial'_\mu v^i + i\dot{x}^\mu\hat{U}F\partial'_\mu v - n^\mu\omega\mathcal{J}_\mu - i\dot{x}^\mu\tilde{\omega}\mathcal{J}_\mu - i\dot{x}^\mu Q\partial'_\mu v = 0 \ , \tag{5.17}$$

with $\mathcal{J}_\mu$, Q as in (4.20a,b,c). Hence eliminating Q from (4.21b)

$$\begin{aligned} &(U - \hat{U})Gv = \omega g_v^D + \tilde{\omega}g_{Gv} \ , \\ &(\beta_i^A + \tilde{s}_i)v^i = \ -\hat{U}(Fv) + (W + \omega)g_{Gv} + (\tilde{W} + \tilde{\omega})g_v^D \ . \end{aligned} \tag{5.18}$$

In a similar fashion the Weyl consistency conditions arising from $[\Delta^W_\sigma, \Delta^W_{\sigma'}] = 0$ require additional relations involving the various coefficients appearing in the boundary terms in (5.14) where the essential behaviour under local scale transformations is given by (2.22). The commutators in (4.25) now give as well

$$[\Delta^W_\sigma, \int dv\, \sigma' \big(\tfrac{1}{2}\beta^\Phi R - UT'\big)\cdot \frac{\delta}{\delta T}] - (\sigma \leftrightarrow \sigma') \sim \int ds\, \xi_\mu\, n^\mu \theta \cdot \frac{\delta}{\delta \hat{T}} \ ,$$
$$[\int dv\, \partial_\mu \sigma\, i\epsilon^\mu{}_\nu \tilde{s}_i \cdot \frac{\delta}{\delta V_{\nu i}}, \int dv\, \sigma' \big(UT'\big)\cdot \frac{\delta}{\delta T}] - (\sigma \leftrightarrow \sigma') \sim \int ds\, \xi_\mu\, i\dot{x}^\mu (U\tilde{s})\cdot \frac{\delta}{\delta \hat{T}} \ ,$$

and there is also an additional piece on the r.h.s. of (4.27). From the $\hat{\Phi}$ dependent terms appearing in (5.14) it is necessary that, like (4.26),

$$\hat{\Omega} = -\hat{U}\partial \ , \tag{5.19}$$

giving from (5.15)

$$B^{\hat{\Phi}} = \hat{\Delta}\hat{\Phi} + \hat{\theta} \ , \qquad \hat{\Delta} = (d - \hat{U})\partial \ . \tag{5.20}$$

Using (5.19) and collecting all the contributions proportional to $n^\mu \xi_\mu$ and $\dot{x}^\mu \xi_\mu$ in the consistency relations then gives

$$\theta - \hat{\theta} + (U - \hat{U})s + \omega(\beta^g + g_{\tilde{s}}) - \tilde{\omega} g_s + \hat{U}\partial\epsilon + \mathcal{D}_\beta \epsilon = 0 \ , \tag{5.21a}$$
$$(\tilde{W} + \tilde{\omega})(\beta^g + g_{\tilde{s}}) - (W + \omega)g_s + \hat{U}(\beta^A + \partial\tilde{\epsilon} + \tilde{s}) + \mathcal{D}_\beta \tilde{\epsilon} = 0 \ , \tag{5.21b}$$

where now

$$\mathcal{D}_\beta = \sum_{h=G,B,A} \beta^h \cdot \frac{\partial}{\partial h} \ .$$

By using (5.18), and noting that $\mathcal{D}_\beta \epsilon = \mathcal{D}_B \epsilon - \mathcal{L}_d \epsilon$ and similarly for $\tilde{\epsilon}$, with the definitions (5.15) this becomes

$$B^{\hat{\Phi}} - B^\Phi = \omega B^g + \mathcal{D}_B \epsilon + \hat{\Delta}(\hat{\Phi} - \Phi - \epsilon) \ , \tag{5.22a}$$
$$(d - \hat{U})B^A = (\tilde{W} + \tilde{\omega})B^g + \mathcal{D}_B \tilde{\epsilon} \ . \tag{5.22b}$$

(5.22a) is the direct analogue of (2.23) for this σ model. These relations appear as a direct extension of the Curci-Paffuti relation (4.32) and also (4.34) to renormalisable non linear σ models with a boundary, for an alternative discussion see [32].

The one loop results [30] for β^A are

$$\beta^A_i = 2(1 - F^2)^{-1\, jk}\big(\nabla_j F_{ik} - F_i{}^\ell F_k{}^m H_{m\ell j}\big) \ , \tag{5.23}$$

and then from the appendix or by consistency with (5.18) and (5.21b), which to this order requires just $\hat{U}\beta^A = 0$,

$$\hat{U}w = 2(1 - F^2)^{-1\, ji}\big(\nabla_i w_j - F_{ik} H^{k\ell}{}_j w_\ell\big) \ , \quad \tilde{\omega} = \tilde{\epsilon} = 0 \ ,$$
$$\omega \delta g = (1 - F^2)^{-1\, ik} F_k{}^j \delta B_{ij} - \tfrac{1}{2}\Big(\frac{1 + F^2}{1 - F^2}\Big)^{ij} \delta G_{ij} \ , \tag{5.24}$$

with indices raised and lowered with the metric G_{ij}.

For local Weyl invariance then the new conditions which arise in addition to (4.32) for σ models with boundary are from (4.11), (4.14) and (5.14)

$$B_i^A = 0\ , \qquad \hat{\Phi} = \Phi + \epsilon\ , \qquad \hat{\Delta}\hat{T} - \hat{T} = 0\ . \tag{5.25}$$

Clearly (5.22b) provides a linear relation between B_i^A, B_{ij}^G and B_{ij}^B, whose significance is not immediately clear, while (5.22a) shows that subject to (5.25) and $B^g = 0$ $B^{\hat{\Phi}} = B^{\Phi}$.

6. Conclusion

In this paper we have endeavoured to define a local renormalisation group equation expressing the response to a local Weyl rescaling of the metric and then show how non trivial consistency relations may be derived from the commutativity of two different rescalings. To achieve this it was essential to have a framework in which local composite operators are well defined. Thus any divergence present in correlation functions of products of composite operators is required to be cancelled by appropriate counterterms and these determine the extra terms in the local renormalisation group equation beyond those given by conventional β functions. Following similar ideas Shore has recently shown [33] how the renormalisation group equation for the Wilson coefficients appearing in the operator product expansion may be simply derived. An alternative earlier approach by Shore [34] to achieve a local renormalisation group equation, in which the renormalisation mass scale μ is made x dependent, failed to give finite equations.

The discussion here has been confined to renormalisable field theories where power counting provides a strong constraint on the form of the additional terms present in the local renormalisation group equation. It would be natural to attempt to extend some of the present results to the more general Wilson approach where the renormalisation flow of arbitrary field theories under variations of the cut off is described. In this framework a discussion of the consequences of conformal invariance, with a flat space background, has been given by Schäfer [35] and a geometric description of a local renormalisation group flow has been provided by Periwal [36]. Such an extension would be potentially important for applications to string theory which may be regarded as defined by the conformally invariant fixed points in the space of two dimensional field theories. Various [37] authors have applied the Wilson renormalisation flow equations to string models to derive dual S matrix amplitudes, with a flat world sheet, from the conditions for a fixed point in the presence of various backgrounds. However in such approaches, expanding about the trivial fixed point, the full set of gauge invariances of the linearised equations of motion for fields representing massive modes expected from the Virasoro conditions on physical states are not manifest. A natural generalisation would be to impose the conditions for local Weyl invariance, with a curved world sheet, when there are additional auxiliary fields coupled to the curvature, like the dilaton. It is essential that any such extension should maintain explicitly invariance under general reparameterisations of the two dimensional world sheet so as to ensure that at any fixed point complete conformal invariance is recovered.

For four dimensional field theories the results of section 3 are more complicated and in general sensitive to lower dimension operators. However (3.10a) or equivalently

$$8\,\partial_i\tilde{\beta}_b = \chi^g_{ij} + (\partial_i w_j - \partial_j w_i)\beta^j\ , \tag{6.1}$$

is valid in arbitrary renormalisable field theories. Although χ^g_{ij} has not been shown to be positive, except at lowest order in perturbative calculations, the relation (6.1) provides non trivial conditions on the form of β functions such as that for Yukawa couplings at two loops [14].

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Appendix

In this appendix the relationship of the discussion of non linear σ models in sections 4 and 5 with the background field formalism, which provides the most straightforward method for calculating β functions and other related quantities such as appear in the essential equations (4.14) and (5.14), is considered. In particular it is crucial to verify that no boundary counterterms other than of the form prescribed by (5.1) are necessary in a consistent renormalisation treatment.

As usual in the covariant background field formalism applied to σ models the quantum field $\phi^i(x)$, which are coordinates on the target manifold $\mathcal{M}$, is expanded about a fixed background $\varphi^i(x)$ by defining a geodesic path, as determined by the metric G_{ij}, $\phi^i_t(x)$, $0 \leq t \leq 1$, from $\varphi^i(x)$ to $\phi^i(x)$ and then expressing the functional integral in terms of the field $\xi^i(x) = \dot{\phi}^i_0(x) \in T\mathcal{M}_{\varphi(x)}$. The action $S(\phi, \lambda) + \hat{S}(\phi, \hat{\lambda})$ is then expanded in powers of ξ which ensures automatic invariance under reparameterisations of φ. For the part involving an integral over the two dimensional space the result has the form $S_c + S_1 + S_2 + \ldots$ where $S_c = S(\varphi, \lambda)$ and $S_n = \mathcal{O}(\xi^n)$. To first order from (5.1), setting $\Phi = 0$, and also later $\hat{\Phi} = 0$, for simplicity

$$\begin{aligned} S_1 &= \int dv \left(G_{ij} \partial^\mu \varphi^i \tilde{\nabla}_\mu \xi^j - A_{\mu ij} \partial^\mu \varphi^i \, \xi^j + T'_i \, \xi^i \right) , \\ \tilde{\nabla}_\mu \xi^i &= \nabla_\mu \xi^i + i\epsilon_\mu{}^\nu \partial_\nu \varphi^k H^i{}_{kj} \, \xi^j \ , \qquad \nabla_\mu \xi^i = \partial_\mu \xi^i + \partial_\mu \varphi^k \Gamma^i_{kj} \, \xi^j \ , \end{aligned} \tag{A.1}$$

where $A_{\mu ij}$, T'_i are as in (4.4) and like G_{ij}, $H^i{}_{kj}$ and Γ^i_{kj}, which is the Christoffel connection formed from G_{ij}, are evaluated at φ. In obtaining S_1 from the expansion of $S(\phi, \lambda)$ a total divergence $\nabla_\mu \rho^\mu$, $\rho^\mu = i\epsilon^{\mu\nu} B_{ij} \xi^i \partial_\nu \varphi^j + V^\mu{}_i \xi^i$ has been discarded. The boundary term given by (5.1) may also be similarly expanded, with $\hat{S}_c = \hat{S}(\varphi, \hat{\lambda})$, and to first order in ξ, including the piece $-n^\mu \rho_\mu$ which remains after deriving (A.1), we obtain

$$\hat{S}_1 = \int ds \left(-i\dot{x}^\mu \partial_\mu \varphi^i F_{ij} \, \xi^j + \hat{T}'_i \, \xi^i \right) , \tag{A.2}$$

with the definitions in (5.6). In the form (A.1) and (A.2) S_1 and $\hat{S}_1$ are manifestly invariant under the gauge symmetries (4.3) and (5.4), (5.5) and since higher order terms in the expansion in ξ may be generated iteratively from S_1, $\hat{S}_1$ [38] this is therefore true for S_n, $\hat{S}_n$ for any n.

The requirement that $S_1 + \hat{S}_1$ vanishes for arbitrary ξ on the boundary of the two dimensional space provides a boundary condition for φ. In order that the background field technique gives a consistent method for calculating the necessary counterterms for finiteness of the full quantum field theory it is necessary that any boundary conditions should be imposed on the essential quantum field $\phi^i(\varphi, \xi)$ so that invariance under the shift symmetry $\delta\varphi^i = \eta^i$, $\delta\xi^i = N^i{}_j(\varphi, \xi)\eta^j$, leaving ϕ invariant, is maintained [39]. Thus we require

$$n^\mu G_{ij}(\phi) \partial_\mu \phi^j = i\dot{x}^\mu F_{ij}(\phi) \partial_\mu \phi^j + \hat{T}'_i(\phi) \ . \tag{A.3}$$

This may be expanded by assuming the background field φ^i also obeys the same boundary condition and then the associated boundary condition for ξ is determined to be

$$n^\mu G_{ij}\nabla_\mu\xi^j = i\dot{x}^\mu F_{ij}\nabla_\mu\xi^j + i\dot{x}^\mu\partial_\mu\varphi^k F_{ik;j}\,\xi^j + \hat{T}'_{i;j}\,\xi^j + \mathcal{O}(\xi^2)\ , \tag{A.4}$$

where the coefficients are evaluated at φ and the covariant derivatives are defined by the connection $\Gamma^i_{kj}(\varphi)$. (A.4) then determines the necessary boundary condition on the Greens functions associated with the internal lines of any Feynman graph while (A.3) for $\phi\to\varphi$ shows how any divergences involving $n^\mu\partial_\mu\varphi$ on the boundary may still be cancelled by counterterms with the structure of (5.1).

At second order in ξ standard calculations show that [13]

$$\begin{aligned}
S_2 &= \int dv\, \tfrac12\big(G_{ij}D^\mu\xi^i D_\mu\xi^j + X_{ij}\xi^i\xi^j\big)\ , \qquad D_\mu\xi^i = \tilde\nabla_\mu\xi^i - \tfrac12 A_\mu{}^i{}_j\xi^j\ ,\\
X_{ij} &= \partial^\mu\varphi^k\partial_\mu\varphi^\ell\big(R_{kij\ell} + H_{kim}H^m{}_{\ell j}\big) + i\epsilon^{\mu\nu}\partial_\mu\varphi^k\partial_\nu\varphi^\ell H_{k\ell(i;j)}\\
&\quad - \partial^\mu\varphi^k\big(G_{k\ell}\partial'_\mu\Gamma^\ell_{ij} + A_{\mu k(i;j)} + i\epsilon_\mu{}^\nu H_{k(i}{}^\ell A_{\nu j)\ell}\big) - \tfrac14 A_{\mu ki}A^{\mu k}{}_j + T'_{(i;j)}\ ,\\
\hat S_2 &= \int ds\, \tfrac12\big(-i\dot x^\mu F_{ij}\nabla_\mu\xi^i\,\xi^j + Q_{ij}\xi^i\xi^j\big)\ , \qquad Q_{ij} = \hat T'_{(i;j)} - i\dot x^\mu\partial_\mu\varphi^k F_{k(i;j)}\ .
\end{aligned} \tag{A.5}$$

With this notation the boundary condition (A.4) becomes, using (5.7) and neglecting $\mathcal{O}(\xi^2)$ corrections,

$$n^\mu G_{ij}D_\mu\xi^j = \tfrac12 i\dot x^\mu\big(\nabla_\mu(F_{ij}\xi^j) + F_{ij}\nabla_\mu\xi^j\big) + Q_{ij}\xi^j\ . \tag{A.6}$$

By introducing a tangent frame basis

$$\begin{aligned}
G_{ij} = e_{ai}e_{aj}\ ,\quad \xi_a = e_{ai}\xi^i\ ,\quad e_{ai}D_\mu\xi^i = \mathcal{D}_\mu\xi_a + s_{\mu ab}\xi^b\ ,\\
\mathcal{D}_\mu\xi_a = \partial_\mu\xi_a + \mathcal{A}_{\mu ab}\xi_b\ ,\quad \mathcal{A}_{\mu ab} = -\mathcal{A}_{\mu ba}\ ,\quad s_{\mu ab} = \tfrac12 e_{ai}e_{bj}\partial'_\mu G^{ij}\ ,
\end{aligned} \tag{A.7}$$

S_2 in (A.5) becomes

$$\begin{aligned}
S_2 &= \int dv\, \tfrac12\big(\mathcal{D}^\mu\xi_a\mathcal{D}_\mu\xi_a + \mathcal{X}_{ab}\xi_a\xi_b + \nabla_\mu(s^\mu{}_{ab}\xi_a\xi_b)\big)\ ,\\
\mathcal{X}_{ab} &= e_a{}^i e_b{}^j X_{ij} + (s_\mu s^\mu - \nabla_\mu s^\mu - [\mathcal{A}_\mu, s^\mu])_{ab}\ .
\end{aligned} \tag{A.8}$$

From (A.8) and $\hat S_2$ in (A.5) and using the condition (A.6) we find

$$S_2 + \hat S_2 = \int dv\, \tfrac12\,\xi_a(\Delta\xi)_a\ , \qquad \Delta = -\frac{1}{\sqrt{\gamma}}\mathcal{D}_\mu\sqrt{\gamma}\gamma^{\mu\nu}\mathcal{D}_\nu + \mathcal{X}\ , \tag{A.9}$$

where Δ is a symmetric elliptic operator, with the associated boundary condition from (A.6)

$$\begin{aligned}
n^\mu\mathcal{D}_\mu\xi_a &= \tfrac12 i\dot x^\mu\big(\mathcal{D}_\mu(F_{ab}\xi_b) + F_{ab}\mathcal{D}_\mu\xi_b\big) - \psi_{ab}\,\xi_b\ ,\\
\psi_{ab} = \psi_{ba} &= -e_a{}^i e_b{}^j Q_{ij} + n^\mu s_{\mu ab} - \tfrac14 i\dot x^\mu\{A_\mu, F\}_{ab} + \tfrac12 n^\mu\{\partial_\mu\varphi H, F\}_{ab}\ ,
\end{aligned} \tag{A.10}$$

where $(\partial_\mu\varphi H)_{ab} = e_a{}^i e_b{}^j \partial_\mu\varphi^k H_{ikj}$. The r.h.s. of (A.10) defines a symmetric operator on the boundary.

At one loop the contribution to the vacuum energy resulting from (A.9) is $-\frac{1}{2}\ln\det\Delta$. With an appropriate regularisation of the functional determinant it is possible to show that (for $\ell = 4\pi$) [40]

$$\begin{aligned}
\Delta^W_\sigma S_0^{(1)} &= \int dv\,\sigma\big\{\tfrac{1}{6}DR - \mathrm{tr}\,\mathcal{X}\big\} + \int ds\,\sigma\,\big\{\tfrac{1}{3}DK + 2\,\mathrm{tr}\big((1-F^2)^{-1}\psi\big)\,\big\} \\
&\quad + \int ds\,\partial_\mu\sigma n^\mu\,\mathrm{tr}\big(F^{-1}\tanh^{-1}F - \tfrac{1}{2}\big) \\
&= \int dv\,\Big\{\sigma\big(\tfrac{1}{6}DR - G^{ij}X_{ij} - \tfrac{1}{4}G^{ik}G^{j\ell}\partial'_\mu G_{ij}\,\partial'^\mu G_{k\ell}\big) + \partial_\mu\sigma\,\tfrac{1}{2}G^{ij}\partial'_\mu G_{ij}\Big\} \\
&\quad + \int ds\,\Big\{\partial_\mu\sigma n^\mu\,\mathrm{tr}\big(F^{-1}\tanh^{-1}F - \tfrac{1}{2}\big) + \sigma\big(\tfrac{1}{3}DK + n^\mu\,\tfrac{1}{2}G^{ij}\partial'_\mu G_{ij} \\
&\qquad + 2(1-F^2)^{-1\,ij}\big(i\dot{x}^\mu\partial_\mu\varphi^k\nabla_j F_{ki} - \nabla_j\hat{T}'_i\big) \\
&\qquad + n^\mu(1-F^2)^{-1}{}_j{}^i\big(2\partial_\mu\varphi^\ell F_i{}^k H_{k\ell}{}^j + i\epsilon_\mu{}^\nu F_{ik}A_\nu{}^{kj} + \partial'_\mu G_{ik}\,G^{kj}\big)\Big\}\ .
\end{aligned} \tag{A.11}$$

Using the boundary condition (A.3), for $\phi \to \varphi$, it is easy to see that this result (A.11) is in accord with (5.23) for β_i^A and also with (5.24) for the other renormalisation group functions defined in (5.14). In addition $\hat{\theta} = \frac{1}{3}D$ and (A.11) determines ϵ at one loop. Furthermore the non boundary terms are in agreement with (4.44).

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