B6b **Variational Principles: Example Sheet 2** Easter term 2023

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1. Using the Lagrange multiplier method, write down the Euler-Lagrange equations associated to the problem of minimising the functional

$$I[\psi] = \int_{-\infty}^{+\infty} \left(\psi'^2 + x^2\psi^2\right) dx$$

subject to the normalization condition $\int \psi^2 dx = 1$. Given that $x\psi(x)^2 \to 0$ as $x \to \pm\infty$, show that

$$I[\psi] = 1 + \int_{-\infty}^{+\infty} (\psi' + x\psi)^2 dx \,,$$

and hence deduce that $I \geq 1$. Show that equality holds for a function ψ that you should give explicitly. Verify that it satisfies the Euler-Lagrange equation for an appropriate value of the Lagrange multiplier.

2. Let $\mathbf{x}(t) \in \mathbb{R}^3$ be a curve which is constrained to lie on the sphere $S^2 = \{\mathbf{x} : |\mathbf{x}| = 1\}$. Use the Lagrange multiplier function formalism to obtain the following Euler-Lagrange equation

$$\ddot{\mathbf{x}} + |\dot{\mathbf{x}}|^2\mathbf{x} = \mathbf{0}$$

for the problem of minimising $I[\mathbf{x}] = \int |\dot{\mathbf{x}}|^2 dt$ amongst curves satisfying the constraint $\mathbf{x}(t) \in S^2$. Show that the solutions of the Euler-Lagrange equation lie on a plane through the origin (i.e. that they are great circles.)

3. Obtain the Euler-Lagrange equations associated with the functionals

(i) $I[u] = \int [\frac{1}{2}u_t^2 - F(u_x)]dxdt\,,$ (ii) $I[u] = \int [|\nabla u|^2 + e^{2u}]dxdy$.

4. Show that:

(i) x^2/y is convex on the upper half plane $(x, y) : y > 0$.

(ii) the function $F(x, y) = yf(x/y)$ (called the "perspective" of f) is convex on $(x, y) : y > 0$ if $f(x)$ is convex [*Hint: after introducing* $t \in (0, 1)$ *use the new variable* $s = \frac{ty'}{(1-t)y+ty'}$]. Now, assuming f to be twice differentiable, verify convexity of F by computing its Hessian matrix.

5. Find the Legendre transform of $f(x) = e^x$, (giving its domain also). Find the Legendre transform of $f(x) = a^{-1}x^a, a > 1$ defined on $x > 0$, and hence deduce Young's inequality

$$xy \;\leq\; \frac{x^a}{a} + \frac{y^b}{b}\,, \qquad \frac{1}{a} + \frac{1}{b} = 1\,.$$

6. For an ideal gas, the internal energy $U = U(S, V)$ as a function of entropy and volume is

$$U = U_0 + \alpha nRT_0\Big[\big(\frac{V_0}{V}\big)^{\frac{1}{\alpha}} e^{\frac{S-S_0}{\alpha nR}} - 1\Big]$$

for some constants $U_0, T_0, V_0, S_0, \alpha, n, R$. Calculate the Helmholtz free energy $F = F(T, V)$ defined by $F(T, V) = \min_S(U(S, V) - TS)$.

7. A particle of mass m is constrained to roll on the inside of a smooth upturned hemispherical bowl of radius a. The Lagrangian describing the motion is

$$L = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2(\sin^2\theta)\dot{\phi}^2 + mga\cos\theta\,,$$

where g is the acceleration due to gravity, and θ and ϕ are the usual spherical angles (with θ measured relative to the downward vertical). Find two constants of the motion.

Find the two momenta p_θ and p_ϕ and hence the particle's Hamiltonian. What do Hamilton's equations become in this case?

8. Hamilton's Principle is applicable to the *relativistic* dynamics of a charged particle in an electromagnetic field. The appropriate choice of Lagrangian $L[\mathbf{x}(t), \dot{\mathbf{x}}(t), t]$ for a particle of rest-mass m and charge q in a given electric potential $\phi(t, \mathbf{x})$ and magnetic vector potential $\mathbf{A}(t, \mathbf{x})$ is

$$L = -mc^2\sqrt{1 - |\mathbf{v}|^2/c^2} - q\phi + q\mathbf{v} \cdot \mathbf{A} \,,$$

where $\mathbf{v} = \dot{\mathbf{x}}(t)$. Verify that the Euler-Lagrange equations yield the equation of motion

$$\frac{d}{dt}(m_0\gamma\mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \,, \qquad \gamma = (1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}} \,,$$

where $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ (the electric field) and $\mathbf{B} = \nabla \times \mathbf{A}$ (the magnetic field).

9. The mass density $\rho(t, \mathbf{x})$ and velocity field $\mathbf{v}(t, \mathbf{x})$ of a compressible fluid are constrained by conservation of mass to satisfy the continuity equation

$$\dot{\rho} + \nabla \cdot (\rho\mathbf{v}) = 0 \,. \qquad (*)$$

Given that the energy density of the fluid is $u(\rho)$, the action (for inviscid irrotational flow) is

$$S[\rho, \mathbf{v}, \phi] = \int dt \int d^3x \left\{ \frac{1}{2}\rho|\mathbf{v}|^2 - u(\rho) + \phi\left[\dot{\rho} + \nabla \cdot (\rho\mathbf{v})\right] \right\} \,,$$

where $\phi(t, \mathbf{x})$ is a Lagrange multiplier field imposing the continuity condition $(*)$. Find the Euler-Lagrange equations for this action. Show that they imply $\mathbf{v} = \nabla\phi$ (so ϕ is the velocity potential). Given that the fluid pressure $P(t, \mathbf{x})$ satisfies

$$\nabla P = \rho\nabla h(t, \mathbf{x}) \,, \qquad h = u'(\rho) \,,$$

deduce Euler's equation for inviscid irrotational flow:

$$\rho\left[\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\,\mathbf{v}\right] = -\nabla P \,.$$

10. If a curve between points A and B on the unit sphere can be parametrised by the polar angle θ then its length is given by the functional $L[\phi] = \int_A^B (1 + \phi'^2 \sin^2\theta)^{\frac{1}{2}}\, d\theta$. Show that $\delta^2 L$ is positive.

    If the curve can be parametrised by the azimuthal angle ϕ then its length is given by the functional $\tilde{L}[\theta] = \int_A^B (\theta'^2 + \sin^2\theta)^{\frac{1}{2}}\, d\phi$. Why does your result for $L[\phi]$ not imply that $\delta^2\tilde{L}$ is positive?

11. For $F[y] = \int_\alpha^\beta (y'^2 + y^4)dx$ with $y(\alpha) = a$, $y(\beta) = b$, show that $\delta^2 F$ is strictly positive, and hence that any solution of the Euler-Lagrange equation is a local minimum of F. Write down the Euler-Lagrange equation and find its solution for the case $a = b = 0$. Why is this solution a global minimum of F?

12. A function $y(x)$ defined for $0 \leq x \leq 1$ is such that $y(0) = y(1) = 0$. Write down the Euler-Lagrange equation associated to the functional

$$F[y] = \int_0^1 \left(\frac{1}{2}y'^2 + g(y)\right) dx \,,$$

where $g(y)$ is such that $g'(0) = 0$. Show that $y_0(x) = 0$ is a solution. Given that the Euler-Lagrange equation is satisfied, find $\delta^2 F$ and determine the range of values of $g''(0)$ for which it is positive. [*This includes a range of negative values of* $g''(0)$.]